

DOUBLE OBSTACLE PROBLEMS WITH OBSTACLES GIVEN BY NON- C^2 HAMILTON-JACOBI EQUATIONS

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ABSTRACT. We prove optimal regularity for the double obstacle problem when obstacles are given by solutions to Hamilton-Jacobi equations that are not C^2 .

When the Hamilton-Jacobi equation is not C^2 then the standard Bernstein technique fails and we lose the usual semi-concavity estimates. Using a non-homogeneous scaling (different speed in different directions) we develop a new pointwise regularity theory for Hamilton-Jacobi equations at points where the solution touches the obstacle.

A consequence of our result is that C^1 -solutions to the Hamilton-Jacobi equation

$$\pm|\nabla h - a(x)|^2 = \pm 1 \text{ in } B_1, \quad h = f \text{ on } \partial B_1,$$

are in fact $C^{1,\alpha/2}$ provided that $a \in C^\alpha$. This result is optimal and to the authors' best knowledge new.

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1. INTRODUCTION.

1.1. Background. The classical torsion problem modelling the elastic-plastic torsion of a bar can be formulated as follows:

$$\text{Minimize } \int_{\Omega} (|\nabla u|^2 - u) dx$$

in the convex set $K = \{u \in W_0^{1,2}(\Omega); |\nabla u|^2 \leq 1\}$ (see [?], [?], [?], [?]). Brezis and Sibony [?] showed that this problem is equivalent to minimizing the above energy in the set $\tilde{K} = \{u \in W_0^{1,2}(\Omega); u \leq d(x)\}$ where $d(x)$ is the distance function to the boundary.

More generally, one can show that minimizing the Dirichlet energy in $L = \{u \in W^{1,2}(\Omega); |\nabla u|^2 \leq 1 \text{ and } u = f \text{ on } \partial\Omega\}$ is equivalent to minimizing the Dirichlet energy in the set $\tilde{L} = \{u \in W^{1,2}(\Omega); h^- \leq u \leq h^+\}$ where h^{\pm} solves the Hamilton-Jacobi equations

$$\begin{aligned} \pm |\nabla h^{\pm}|^2 &= \pm 1 && \text{in } \Omega, \\ h^{\pm} &= f && \text{on } \partial\Omega, \end{aligned}$$

provided that L and \tilde{L} are non empty, which incidentally are equivalent conditions (see [?] for a proof of this equivalence).

The regularity theory for a minimizer of the Dirichlet energy in \tilde{L} is quite straightforward. Indeed, one may approximate h^+ by the solution h_{ϵ}^+ to

$$\begin{aligned} -\epsilon \Delta h_{\epsilon}^+ + |\nabla h_{\epsilon}^+|^2 &= 1 && \text{in } \Omega, \\ h_{\epsilon}^+ &= f && \text{on } \partial\Omega, \end{aligned}$$

whence the Bernstein technique (see [?] or Lemma 6.2 below) gives an ϵ -independent estimate on the second derivatives from above. In particular, letting $\epsilon \rightarrow 0$ we may deduce that h^+ is semi-concave, or equivalently that the distributional second derivatives of h^+ are bounded from above. Similarly, the second order distributional derivatives of h^- are bounded from below. From here, standard regularity theory for the obstacle problem (such as developed in [?], modified slightly in the present paper to suit our purposes) implies that $u \in C_{\text{loc}}^{1,1}$.

The important step in the above proof is to deduce one-sided estimates on the distributional second derivatives of h^{\pm} . More generally, if u is a minimizer to the Dirichlet energy in \tilde{L} with h^+ being a solution to a Hamilton-Jacobi equation $F(x, h^+, \nabla h^+) = 1$ (h^- solves $-F(x, h^-, \nabla h^-) = -1$) for an $F \in C^2$ satisfying some structural assumptions, then the same technique yields $u \in C^{1,1}$. The important step again is the Bernstein technique where we need to differentiate F twice. See for instance [?] or [?] for variational problems of this

type or [?] with errata [?] for a fully nonlinear gradient constrained problem.

When $F \notin C^2$, the above outline of a regularity proof fails. The existing regularity theory for Hamilton-Jacobi equations is not sufficiently strong for that purpose. There are cases where certain one-sided estimates have been deduced, see [?] and [?]. For example, in [?] and [?] it is assumed that $F \in C^{0,1}$ (besides standard structural assumptions) and it is shown that $h^+(x + x^0) \leq h^+(x^0) + p \cdot x + |x|\sigma(|x|)$ for any p in the super-differential of h^+ at x^0 where σ is some one-sided modulus of continuity over which we have no control.

1.2. Main Result and Ideas. The objective of this paper is to introduce new techniques handling regularity questions for Hamilton-Jacobi equations below the C^2 -threshold. It is noteworthy that the minimization problem in class L with Hamilton-Jacobi equations below C^2 -threshold has applications in micro-magnetics. In view of the equivalence of the minimization problem in class L and \tilde{L} for the special case of gradient constraint $|\nabla u| \leq 1$ (see Background), we study in the present paper the problem in \tilde{L} with variable coefficients that are not Lipschitz.

Our main results are the following theorem and its corollary.

Theorem 1.1. *[Main Theorem] Let u be a minimizer of the Dirichlet energy*

$$\int_{B_1} |\nabla u|^2$$

in the set $\{u \in W^{1,2}; h^- \leq u \leq h^+, u = f \in C^\alpha(\partial B_1) \text{ on } \partial B_1\}$, where h^\pm are viscosity solutions to

$$(1) \quad \begin{aligned} \pm |\nabla h^\pm - a(x)|^2 &= \pm 1 && \text{in } B_1, \\ h^\pm &= f && \text{on } \partial B_1, \end{aligned}$$

with $[a]_{C^\alpha(\overline{B_1})} = A \leq \tilde{C}$ for some $\tilde{C} < +\infty$. Assume furthermore that $u(0) = h^+(0)$. Then

$$(2) \quad \begin{aligned} \text{osc}_{x \in B_r(0)} \left(u(x) - u(0) - \nabla u(0) \cdot x \right) &\leq C(\alpha) \sqrt{A} r^{1+\frac{\alpha}{2}} && \text{if } r \leq A^{\frac{1}{2-\alpha}}, \\ \text{osc}_{x \in B_r(0)} \left(u(x) - u(0) - \nabla u(0) \cdot x \right) &\leq C r^2 && \text{if } r > A^{\frac{1}{2-\alpha}}. \end{aligned}$$

A surprising consequence which may be of immediate interest to the regularity theory for Hamilton Jacobi equations is the following corollary.

Corollary 1.2. *Let h be a C^1 solution to $|\nabla h - a|^2 = 1$. Then $h \in C^{1,\alpha/2}$, provided that $a \in C^\alpha$.*

Proof. Since $h \in C^1$, we have, by uniqueness of solutions to HJ-equations, that if h^\pm solves equations

$$\begin{aligned} \pm |\nabla h^\pm - a(x)|^2 &= \pm 1 && \text{in } B_1, \\ h^\pm &= h && \text{on } \partial B_1, \end{aligned}$$

then $h^+ = h^- = h$. In particular the set

$$K = \{u \in W^{1,2}; h^- \leq u \leq h^+\} = \{h\}.$$

Therefore, h is in a trivial way a minimizer of $\int_{B_1} |\nabla u|^2$ in K . By Theorem 1.1 it follows that $h \in C^{1,\alpha/2}$. \square

The function $F(x, p) = |p - a(x)|^2$ is related to a HJ equation that arises in micro-magnetics, and hence our choice is not completely arbitrary (see [?]). Our method is quite robust, and as such it seems plausible to adapt it to a wide class of Hamilton-Jacobi equations.

The main difficulty, as indicated above, is to develop a strong enough regularity theory for Hamilton-Jacobi equations. In this article, we will not treat Hamilton-Jacobi equations in their full generality. Instead we will stay within the confines of the obstacle problem. This has one great advantage: it is easy to see that $u \in C^{1,\beta}$ for some $\beta > 0$ (Lemma 6.5 and Proposition 2.5). We will therefore get a one-sided estimate of h^+ from below at all points where $u = h^+$. That means that the set $\{u = h^+\}$ does not intersect the singular set of h^\pm ; by the singular set of h^+ we mean the set where h^\pm are not differentiable in the classical sense.

The regularity theory for h^+ is deduced by *inhomogeneous scaling*. There is a slight complication to apply this method to Hamilton-Jacobi equations of our type. In particular, even though h^+ satisfies an elliptic Hamilton-Jacobi equation it scales parabolically and the blow-up limit will solve a parabolic equation. Let us denote

$$h_j(x) = \frac{h^+(r_j x', r_j^{1-\beta} x_n)}{r_j^{1+\beta}}.$$

Then if $h_j \rightarrow h_0$ as $r_j \rightarrow 0$, the function h_0 heuristically solves an equation of the form $|\tilde{\nabla} h_0|^2 - 2\partial_n h_0 = 0$ where $\tilde{\nabla} = (\partial_1, \dots, \partial_{n-1}, 0)$.

Our first goal (Proposition 3.2) is to show that the h_j defined above is indeed bounded. It is here that we use the assumption $u(0) = h^+(0)$, which gives a one-sided estimate from below on h^+ .

Once that is proved we can use the regularity theory for parabolic Hamilton-Jacobi equations to deduce that h^+ satisfies somewhat better one-sided estimates in the x' directions, for example

$h^+(x', 0) \leq h^+(0) + p \cdot x' + C|x'|^{1+\beta+\epsilon}$ where $\epsilon = \epsilon(\beta, \alpha)$ (see Proposition 4.1).

The minimizer u on the other hand scales elliptically, so if

$$u(r_j x) / \sup_{B_{r_j}} |u|$$

converges to u_0 as $r_j \rightarrow 0$, then u_0 is a solution to an obstacle problem with obstacle $h_0 = \lim_{j \rightarrow \infty} h^+(r_j x) / \sup_{B_{r_j}} |u|$. This fact will be used in the proof of our main theorem, Theorem 1.1, in order to show that $\text{osc}_{B_r} |u(x) - \nabla u(0) \cdot x| \leq C|x|^{1+\beta+\epsilon}$. In particular we gain an ϵ in the regularity of u . By carefully keeping track of all the constants we see that this can be iterated to obtain γ -independent $C^{1,\gamma}$ estimates of u for all $\gamma < \alpha/2$. It follows that $u \in C^{1,\alpha/2}$.

The following example pointed out to us by Stefan Müller shows that this is indeed the optimal regularity:

Example 1.3. *If $a(x) = |x_2|^\alpha e_1$, then $u(x_1, x_2) = x_1 + b(x_2)$ is a solution to $|\nabla u - a(x)|^2 = 1$ for*

$$b(x_2) = \int_0^{x_2} \sqrt{2|t|^\alpha - |t|^{2\alpha}} dt.$$

is a solution to $|\nabla u - a(x)|^2 = 1$. Here $a \in C^\alpha$ and $u \in C^{1,\alpha/2}$, but $u \notin C^{1,\beta}$ for any $\beta > \alpha/2$.

The plan of the paper is as follows. In Section 2 we deduce an abstract regularity result for solutions to the double obstacle problem, which we will need later. In the subsequent two sections we show that h^+ remains bounded under parabolic scaling and that h^+ satisfies better one-sided estimates in the x' directions. In the final section we prove our main result that the minimizer $u \in C^{1,\alpha/2}$. Finally we have included a long appendix to remind the reader of some of the theory of viscosity solutions for Hamilton-Jacobi equations. In the Appendix we also deduce simple $C^{1,\beta}$ -estimates for the solution, which will serve as our starting regularity in the strategy described above.

Acknowledgment: We would like to thank Stefan Müller for providing us with Example 1.3.

1.3. Notation. We denote the Euclidean ball $B_r(x) = \{y \in \mathbb{R}^n; |y - x| < r\}$, and we denote $\omega_n := \mathcal{L}^n(B_1)$; in the case that the center is not specified it is assumed to be the origin. When $v \in \mathbb{R}^n$ is a vector we will denote the first $n - 1$ coordinates by $v' := (v_1, v_2, \dots, v_{n-1})$. Similarly, we will use $\tilde{\nabla} := (\partial_1, \partial_2, \dots, \partial_{n-1}, 0)$. Here $\partial_i = \frac{\partial}{\partial x_i}$ denotes

differentiation with respect to the x_i variable. We will often denote differentiation by a subscript $\partial_i u =: u_i$. The unit vector in the i -th coordinate direction will be denoted by e_i . We will also use the seminorm

$$[a]_\alpha = [a]_{C^\alpha(\bar{\Omega})} := \sup_{x, y \in \bar{\Omega}} \frac{|a(x) - a(y)|}{|x - y|^\alpha}.$$

Finally, different instances of the letter C may mean different constants even within one proof or one set of inequalities.

2. REGULARITY FOR DOUBLE OBSTACLE PROBLEMS

In this section we will prove regularity for the two-sided obstacle problem in the context needed later.

It will be convenient to define a class of solutions to double obstacle problems and to fix some notation. We therefore state the following two definitions before we state and prove the main result of this section.

Definition 2.1. *For any continuous function u we define the super-differential (sub-differential) of u at the point $x^0 \in \text{Domain}(u)$ as follows:*

$$S^+(u, x^0) = \left\{ p \in \mathbb{R}^n; \sup_{B_r} (u(x + x^0) - u(x^0) - p \cdot (x - x^0)) \leq o(r) \right\}$$

$$(S^-(u, x^0) = \left\{ p \in \mathbb{R}^n; \inf_{B_r} (u(x + x^0) - u(x^0) - p \cdot (x - x^0)) \geq -o(r) \right\}).$$

Definition 2.2. *We define $\mathcal{C}(R, \sigma, h^\pm)$ as the set of local minimizers u to the Dirichlet energy*

$$\int_{B_R} |\nabla u|^2$$

in the set $K = \{u \in W_{\text{loc}}^{1,2}(B_R); h^- \leq u \leq h^+\}$; here $0 < R \in \mathbb{R}$, $\sigma(r)$ is a “one-sided modulus of continuity” of u and $h^\pm \in C^0(B_R)$ such that $h^+ \geq h^-$.

Furthermore we require that u satisfies the following estimates for each point $x^0 \in \Lambda(u, R) := (\{u = h^+\} \cup \{u = h^-\}) \cap B_R$:

(i) *when $u(x^0) = h^+(x^0)$, then we have the bound from above*

$$\sup_{B_r(x^0)} \inf_{p_{x^0} \in \mathbb{R}^n} (u(x) - p_{x^0} \cdot (x - x^0) - u(x^0)) \leq \sigma(r) \text{ for all } r > 0.$$

(ii) *when $u(x^0) = h^-(x^0)$, then we have the bound from below*

$$\inf_{B_r(x^0)} \sup_{p_{x^0} \in \mathbb{R}^n} (u(x) - p_{x^0} \cdot (x - x^0) - u(x^0)) \geq -\sigma(r) \text{ for all } r > 0.$$

Remark 2.3. It is important that by (i) and (ii), u inherits its “one-sided modulus of continuity” from that of h^\pm .

Proposition 2.4. Let $u \in \mathcal{C}(R, \sigma, h^\pm)$ where $\sigma(r)$ is a one-sided modulus of continuity satisfying the doubling condition

$$(3) \quad \frac{\sigma(Sr)}{\sigma(r)} \leq F(S) \text{ for each } S \geq 1 \text{ and all } r > 0.$$

for each $S \geq 1$. Then if $x^0 \in \Lambda(u, R/2)$ it follows that

$$\operatorname{osc}_{B_r(x^0)} \inf_{p_{x^0} \in \mathbb{R}^n} (u - p_{x^0} \cdot (x - x^0)) \leq C(n, F)\sigma(r).$$

Remark: It is of no importance that the set K has the particular form $K = \{u \in W^{1,2}; h^- \leq u \leq h^+\}$. We also do not need that h^\pm solves any particular Hamilton-Jacobi equation in this Proposition. The important thing is that we have one-sided estimates at points where u is not harmonic. Similarly the boundary values f are of minor importance.

If we consider the function F in (3) to be $F(s) = s^\alpha$ then the Proposition slightly generalizes known regularity results for double obstacle problems.

Proof of Proposition 2.4: We argue by contradiction and assume that there is a sequence u_j of minimizers as in the proposition, with $\sigma = \sigma_j$ (σ_j corresponds to a fixed F) and points $x^j \in \Lambda(u, R)$ (without loss of generality we are going to assume that $x^j = 0$), $p_j \in S^+(u, x)$ or $p_j \in S^-(u, 0)$ as well as $r_j > 0$ such that

$$(4) \quad \operatorname{osc}_{B_1} \inf_{p_j \in \mathbb{R}^n} \frac{u_j(r_j x) - r_j p_j \cdot x}{j\sigma_j(r_j)} = 1.$$

We may also assume that for large j , r_j is the largest such r corresponding to u^j , that is for $p \in S^+(u, 0)$ or $p \in S^-(u, 0)$

$$(5) \quad \operatorname{osc}_{B_1} \frac{u_j(rx) - rp \cdot x}{j\sigma_j(r)} \leq 1$$

for $r \geq r_j$.

Next, we define

$$v^j(x) = \frac{u_j(r_j x) - r_j p_j \cdot x - u_j(0)}{j\sigma(r_j)},$$

where p_j is the vector in equation (4). Then $v^j \in \mathcal{C}(R, \sigma(r_j \cdot)/(j\sigma(r_j)), \tilde{h}_j^\pm)$ where

$$\tilde{h}_j^\pm(x) = \frac{h_j^\pm(r_j x) - r_j p_j \cdot x - u_j(0)}{j\sigma(r_j)}.$$

In particular $\Delta v^j = 0$ in some set $\Omega_j = B_{R/r_j} \setminus \Lambda(v^j, R/r_j)$, and for every point $x^0 \in \Lambda(v^j, R/(2r_j))$ we have the following: if $v^j(x^0) = \tilde{h}_j^+(x^0)$ then (cf. condition (i) in Definition 2.2)

$$(6) \quad \sup_{B_r(x^0)} \inf_{p \in \mathbb{R}^n} (v^j(x) - p \cdot (x - x^0) - v^j(x^0)) \leq \frac{\sigma_j(rr_j)}{j\sigma(r_j)} \leq \frac{F(r)}{j} \rightarrow 0.$$

Moreover, if $v^j(x^0) = \tilde{h}_j^-(x^0)$ then it follows from condition (ii) in Definition 2.2 that

$$(7) \quad \sup_p \inf_{B_r(x^0)} \sup_{p \in \mathbb{R}^n} (v^j(x) - p \cdot (x - x^0) - v^j(x^0)) \geq -\frac{\sigma_j(rr_j)}{j\sigma(r_j)} \leq \frac{F(r)}{j} \rightarrow 0.$$

Let us write $\Lambda(v^j, R/r_j) = \Lambda^+(v^j, R/r_j) \cup \Lambda^-(v^j, R/r_j)$ with (6) satisfied for points in Λ^+ and (7) satisfied in Λ^- .

For some subsequence $v^j \rightarrow v^0$ in L_{loc}^p , where

$$v^0 \in \mathcal{C}(t, 0, \tilde{h}_0^\pm)$$

for all $t < +\infty$; here we have used (6) and (7) which imply that

$$\sigma(r_j r)/(j\sigma(r_j)) \rightarrow 0 \quad \text{as } j \rightarrow \infty,$$

as well as the notation $\tilde{h}_0^\pm = \lim_{j \rightarrow \infty} \tilde{h}_j^\pm$. We will actually show later that this convergence is locally uniform. Furthermore, by equation (6) we have that $S^+(v^0, x^0) \neq \emptyset$ for all $x^0 \in \Lambda^+(v^0, +\infty)$ and

$$v^0(x) \leq \inf_{x^0 \in \Lambda^+} \inf_{p_{x^0}^+ \in S^+(v^0, x^0)} (v^0(x^0) + p_{x^0}^+ \cdot (x - x^0)),$$

and by equation (7) we have that $S^-(v^0, x^0) \neq \emptyset$ for all $x^0 \in \Lambda^-(v^0, +\infty)$ and that

$$v^0(x) \geq \sup_{x^0 \in \Lambda^-} \sup_{p_{x^0}^- \in S^-(v^0, x^0)} (v^0(x^0) + p_{x^0}^- \cdot (x - x^0)).$$

That is, v^0 solves the double obstacle problem with a concave upper obstacle and a convex lower obstacle. It follows that $\Delta v^0 = 0$ in \mathbb{R}^n . By our assumption that the origin is in $\Lambda(v^0, +\infty)$, say $0 \in \Lambda^+(v^0, +\infty)$, it follows that for $p \in S^+(v^0, 0)$

$$\sup_{B_r} (v^0(x) - p \cdot x) = 0$$

for all $r < +\infty$.

It follows from Liouville's Theorem that v^0 is a linear function, but $v^0(0) = 0$ and by construction (cf. equation (4))

$$(8) \quad \text{osc}_{B_1} \inf_{p \in \mathbb{R}^n} (v^j(x) - p \cdot x) = 1.$$

Equation (8) is a contradiction to v^0 being linear, provided that we can show that $v^j \rightarrow v^0$ uniformly.

In order to show that $v^j \rightarrow v^0$ uniformly we notice that the obstacle functions

$$f_j^+ := \inf_{x^0 \in \Lambda_j^+} \inf_{p_{x^0} \in S^+(v^j, x^0)} \left(v^j(x^0) + p_{x^0} \cdot (x - x^0) \right) + \frac{\sigma(|x|r_j)}{j\sigma(r_j)}$$

converge either locally uniformly to $+\infty$ or that locally uniformly $\lim_{j \rightarrow \infty} f_j^+ = f^+$ where f^+ is an affine linear function. Similarly,

$$f_j^- := \sup_{x^0 \in \Lambda_j^-} \sup_{p_{x^0} \in S^-(v^j, x^0)} \left(v^j(x^0) + p_{x^0} \cdot (x - x^0) \right) - \frac{\sigma(|x|r_j)}{j\sigma(r_j)}$$

will converge either locally uniformly to $-\infty$ or locally uniformly $\lim_{j \rightarrow \infty} f_j^- = f^-$ where f^- is an affine linear function.

Since $f_j^- \leq u^j \leq f_j^+$ we have $f_j^- \leq f_j^+$.

We may distinguish three cases:

1) $v^j(\hat{x}^j) = f_j^+(\hat{x}^j)$ for some bounded sequence of points \hat{x}^j and $v^j(\hat{y}^j) = f_j^-(\hat{y}^j)$ for some bounded sequence of points \hat{y}^j ,

2) $v^j(\hat{x}^j) = f_j^+(\hat{x}^j)$ for some bounded sequence of points \hat{x}^j and $v^j > f_j^-$ in B_{R_j} with $R_j \rightarrow +\infty$, or $v^j(\hat{y}^j) = f_j^-(\hat{y}^j)$ for some bounded sequence of points \hat{y}^j and $v^j < f_j^+$ in B_{R_j} with $R_j \rightarrow +\infty$,

3) $f_j^- < v^j < f_j^+$ in B_{R_j} for a sequence $R_j \rightarrow +\infty$.

In the first case it is easy to see that $|f_j^+ - f_j^-| \rightarrow 0$ locally uniformly: if not then $f_j^- \rightarrow a + p \cdot x$ and $f_j^+ \rightarrow b + p \cdot x$ with $b > a$. Also $v^j \rightarrow c + p \cdot x$ in L_{loc}^2 with $b \geq c \geq a$. We may assume that $p = 0$. Now only one of the inequalities $c > a + (b - a)/3$ or $b - (b - a)/3 > c$ may hold; let us for definiteness assume that $b - (b - a)/3 > c$. Then since $v^j \rightarrow c$ in L_{loc}^2 there is for $T := 2 \sup_j \max(|\hat{x}^j|, |\hat{y}^j|)$ and each $\epsilon > 0$ a $j_\epsilon < +\infty$ such that

$$(9) \quad \frac{1}{(2T)^n} \int_{B_{2T}} |(v^j - c)^+|^2 < \epsilon$$

for $j > j_\epsilon$. Also, as f_j^\pm converges uniformly, $f_j^- < c + \epsilon$ for all $j > j_\epsilon$ (provided that j_ϵ has been chosen large enough). In particular $\{v^j > c + \epsilon\} \cap \Lambda_j^- = \emptyset$ when $j > j_\epsilon$. That means that v^j may touch only the upper obstacle in $\{v^j > c + \epsilon\}$. Thus v^j is subharmonic in $\{v^j > c + \epsilon\}$.

From (9) we deduce that there exists $\tilde{T} \in (3T/2, 2T)$ such that

$$(10) \quad \frac{1}{\tilde{T}^{n-1}} \int_{\partial B_{\tilde{T}}} (v^j - c)^+ \leq C\epsilon$$

for some fixed $C \geq 1$. Now we define w^j by the Poisson integral as

$$(11) \quad w^j(x) = \frac{\tilde{T}^2 - |x|^2}{n\omega_n \tilde{T}} \int_{\partial B_{\tilde{T}}} \frac{\max(v^j, c + C\epsilon)}{|x - y|^n} dy.$$

Then $\Delta w^j = 0$ in $B_{\tilde{T}}$ and $w^j \geq v^j$ on $\partial B_{\tilde{T}}$. Since $\Delta v^j \geq 0$ in $\{v^j > c + \epsilon\}$ and $w^j \geq c + C\epsilon$ it follows that $\max(v^j, w^j)$ is subharmonic. But $w^j \geq v^j$ on $\partial B_{\tilde{T}}$ and $\Delta w^j = 0$ so $\max(v^j, w^j) \leq w^j \leq \max(v^j, w^j)$, where the first inequality follows by comparison and the second is trivial. Thus $w^j \geq v^j$. Moreover we know from (11) and (10) that $w^j \leq c + C\epsilon$ in $B_{\tilde{T}}$. If ϵ is small enough we may deduce that in B_T ,

$$v^j \leq c + C\epsilon < b - \frac{b-a}{3} + C\epsilon < b - \frac{b-a}{6} < f_j^+.$$

But that contradicts the condition in 1) that v^j touches the upper obstacle.

In case 2) we may argue similarly as in case 1) to show that for each T , $f_j^+ \geq v^j > f_j^+ - \epsilon$ in B_T if j is large enough, or $f_j^- \leq v^j < f_j^- + \epsilon$ in B_T if j is large enough, respectively.

In case 3) we have that for each $R < +\infty$, v^j is harmonic in B_R for large j , which implies uniform convergence in C_{loc}^2 . The Proposition follows. \square

Using Lemma 6.5 we can derive a simple Corollary that states some preliminary regularity for solutions to the obstacle problem.

Corollary 2.5. *Let u be a minimizer to the two-sided obstacle problem as above, i.e. u is the minimizer of*

$$\int_{B_R} |\nabla u|^2$$

in the set $\{u \in W^{1,2}; h^- \leq u \leq h^+\}$ and h^\pm are solutions to $\pm|\nabla h^\pm - a|^2 = \pm 1$ where $[a]_\alpha \leq A$. Then if $x^0 \in B_{R/2}$ and $u(x^0) = h^+(x^0)$ (or $u(x^0) = h^-(x^0)$) and p_{x^0} is in the super(sub)-differential of u at x^0 , it follows that

$$(12) \quad \text{osc}_{B_r} \left(u(x + x^0) - p_{x^0} \cdot (x - x^0) \right) \leq C(n) A^{1/(2+\alpha)} r^{1+\alpha/(2+\alpha)}$$

for $r \leq CA^{1/2} R^{\frac{2+\alpha}{2}}$ and that

$$(13) \quad \text{osc}_{B_r} \left(u(x + x^0) - p_{x^0} \cdot (x - x^0) \right) \leq \frac{C}{R} r^2$$

for $r \geq CA^{1/2}R^{\frac{2+\alpha}{2}}$.

In particular, $u \in C^{1, \frac{\alpha}{2+\alpha}}(\overline{B_{R/2}})$.

Proof: Using Lemma 6.5 we see that u satisfies the assumptions of Proposition 2.4 with σ defined by

$$\sigma(r) = CA^{1/(2+\alpha)}r^{1+\alpha/(2+\alpha)}$$

for $r \leq CA^{1/2}R^{(2+\alpha)/2}$, and

$$\sigma(r) = C\frac{r^2}{R},$$

for $r \geq CA^{1/2}R^{(2+\alpha)/2}$.

It follows then by a standard method that $u \in C^{1, \frac{\alpha}{2+\alpha}}$: from (12) it follows that $\nabla u(x^0)$ is well defined. Whitney's extension theorem implies that we can extend $u|_{\text{supp}(\Delta u)}$ to a function \hat{u} defined in $B_{R/2}$ where $\nabla \hat{u}$ is C^α with modulus of continuity

$$\sigma(r) = \begin{cases} C(n)A^{1/(2+\alpha)}r^{\alpha/(2+\alpha)} & \text{for } r \leq CA^{1/2}R^{\frac{2+\alpha}{2}}, \\ \frac{C}{R}r & \text{for } r > CA^{1/2}R^{\frac{2+\alpha}{2}}. \end{cases}$$

In particular if $x, y \in \text{supp}(\Delta u)$ then $|\nabla u(x) - \nabla u(y)| \leq \sigma(|x - y|)$.

Also notice that if $\text{dist}(x^0, \text{supp}(\Delta u)) = |x^0 - y| = \kappa$ where $y \in \text{supp}(\Delta u)$ then $\Delta u = 0$ in $B_\kappa(x^0)$ and the supremum of $|u(x) - u(y) - \nabla u(y) \cdot (x - y)|$ in $B_\kappa(x^0)$ can be estimated by $\sigma(2\kappa)$. Standard regularity theory for harmonic functions implies that $|\nabla u(x^0) - \nabla u(y)| \leq C\sigma(2\kappa)$. In particular it follows that if $x \in B_{R/2} \setminus \text{supp}(\Delta u)$ and $y \in B_{R/2} \cap \text{supp}(\Delta u)$ then $|\nabla u(x) - \nabla u(y)| \leq C\sigma(2|x - y|)$.

If both $y, z \in B_{R/2} \setminus \text{supp}(\Delta u)$ and

$$|y - z| < \frac{1}{2} \max(\text{dist}(y, \text{supp}(\Delta u)), \text{dist}(z, \text{supp}(\Delta u))) = \kappa,$$

then u is harmonic in $B_\kappa(y)$ or in $B_\kappa(z)$. For the sake of definiteness we will assume that u is harmonic in $B_\kappa(y)$. Also $\sup_{B_\kappa(y)} |u(x) - u(y) - \nabla u(y) \cdot (x - y)| \leq \sigma(2\kappa)$. Standard regularity theory for harmonic functions implies that

$$|\nabla u(y) - \nabla u(z)| \leq C\sigma(2\kappa)|y - z|/\kappa \leq C\sigma(|y - z|).$$

Finally, if $y, z \in B_{R/2} \setminus \text{supp}(\Delta u)$ and

$$|y - z| \geq \frac{1}{2} \max(\text{dist}(y, \text{supp}(\Delta u)), \text{dist}(z, \text{supp}(\Delta u))) = \kappa,$$

then we may combine the first two estimates and deduce that

$$|\nabla u(y) - \nabla u(z)| \leq C\sigma(|y - z|).$$

The Corollary follows. \square

3. IMPROVED REGULARITY IN THE NON-DEGENERATE DIRECTION

In this section we will discuss the behavior of a viscosity solution in the non-degenerate direction. To motivate this discussion and our terminology, let us consider a simple elliptic rescaling of $|\nabla h^+ - a|^2 = 1$: if h^+ is differentiable at the origin, $h^+(0) = |\nabla h^+(0)| = 0$, $a(x) = e_n + b(x)$, $|b(0)| = 0$ and

$$h_j(x) = \frac{h^+(r_j x)}{\sup_{B_{r_j}} |h^+|}$$

converges to h_0 as $r_j \rightarrow 0$. If we rescale we see that

$$\begin{aligned} 0 = |\nabla h_j|^2 & \frac{\sup_{B_{r_j}} |h^+|}{r_j} - 2 \frac{\partial h_j}{\partial x_n} - 2b(r_j x) \cdot \nabla h_j \\ & + |b(r_j x)|^2 \frac{r_j}{\sup_{B_{r_j}} |h^+|} + 2b_n(r_j x) \frac{r_j}{\sup_{B_{r_j}} |h^+|}, \end{aligned}$$

and—at least heuristically—

$$\frac{\partial h_0}{\partial x_n} = 0,$$

provided that

$$|b(r_j x)|^2 \frac{r_j}{\sup_{B_{r_j}} |h^+|} + 2b_n(r_j x) \frac{r_j}{\sup_{B_{r_j}} |h^+|} \rightarrow 0, j \rightarrow \infty.$$

Therefore the Hamilton-Jacobi equation is degenerate in all directions except one. Thus we might expect that the oscillation of $h^+(x', x_n)$ along lines $\{(x', s); |s| \leq r\}$ with fixed x' should be of lower order compared to $\text{osc}_{B_r} h^+$. In the next Proposition we will show that is indeed the case, if we have a good estimate on h^+ from below.

The proof, and even the statement, is quite long and not very easy to read. Therefore we will try to explain the general idea before we start.

The idea of the proof is that h^+ scales like $h^+(rx', r^{1-\beta}x_n)/r^{1+\beta}$, whenever $h^+(0) = |\nabla h^+(0)| = 0$ (notice that if $u(0) = h^+(0)$ then ∇h^+ is defined at the origin). It is therefore a natural assumption that $h^+(rx', r^{1-\beta}x_n)/r^{1+\beta}$ will be bounded if $u \in C^{1,\beta}$ and h^+ has one-sided $C^{1+\beta}$ -estimates. We will prove this boundedness (or a slightly refined version of it) by blow-up and a contradiction argument. Most of the proof consists of technical estimates of different terms in the equation for the rescaled h^+ . The idea is simple though. If the supremum of the rescaled h^+ goes to infinity then we can find another rescaling (called h_j in the Proposition) that respects the natural scaling of the

Hamilton-Jacobi equation (Claim 1, below) and is bounded. This new rescaling of h^+ has the nice property that it is worse than a $C^{1,\beta}$ scaling in the x' directions. This will imply that $h_j \rightarrow h_0$ where h_0 is independent of the x' directions, at least at $x_n = 0$ (Claim 2 in the Proposition). The remainder of the proof consists in verifying that h_0 also satisfies a good Hamilton-Jacobi equation that will allow us to conclude that $h^0 \equiv 0$ which in turn will imply a contradiction to the fact that $h_0(0) = 0$ and $\sup_{B_1} |h_0| = 1$.

Remark 3.1. *In the sequel we may assume that $A > 0$. The case $A = 0$ may then be handled by approximating $0 < A_j \rightarrow 0$.*

We will denote

$$(14) \quad K(r) = \left\{ x; \left(\frac{x'}{r}, \frac{A^{\frac{1-\beta}{2}} x_n}{r^{1-\beta}} \right) \in B_1 \right\}.$$

Keeping this definition in mind, let us now state the main result in this section.

Proposition 3.2. *Let h^+ be a viscosity solution to $|\nabla h^+ - a|^2 = 1$ in B_R such that $h^+(0) = 0$. Assume furthermore*

(a1) *that h^+ satisfies one-sided $C^{1,\beta}$ estimates:*

$$(15) \quad \sup_{B_r(x^0)} \left(h^+(x) - h^+(x^0) - p \cdot (x - x^0) \right) \leq CA^{\frac{1-\beta}{2}} r^{1+\beta}$$

for every x^0 and every p in the super-differential of h^+ at x^0 ,

(a2) *that*

$$(16) \quad h^+(x', x_n) \leq h^+(0, x_n) + p \cdot x' + CA^{\frac{1-\beta}{2}} |x'|^{1+\beta}$$

for each fixed x_n ,

(a3) *that*

$$(17) \quad h^+(x) \geq -CA^{\frac{1-\beta}{2}} |x|^{1+\beta},$$

(a4) *that $[a]_{C^\alpha} \leq A$ where A satisfies $A \leq \tilde{C}$ for some \tilde{C} ,*

(a5) *that $a(x) = e_n + b(x)$ where $|b(0)| = 0$,*

(a6) *and that $\alpha/(2 + \alpha) \leq \beta \leq \alpha/2$.*

Then, for $r \leq \min(A^{1/2} R^{\frac{1}{1-\beta}}, R)$,

$$\sup_{K(r)} |h^+(x) + g(x_n)| \leq CA^{\frac{1-\beta}{2}} r^{1+\beta},$$

where

$$(18) \quad g(0) = g'(0) = 0, \quad g'(t) = -e_n \cdot b(0', t),$$

and $K(r)$ is as defined in (14).

Remark 3.3. The condition $r \leq \min(A^{1/2}R^{\frac{1}{1-\beta}}, R)$ assures that $K(r) \subset B_R$ but is otherwise not used in the proof.

Remark 3.4. The assumption (a5) may always be satisfied scaling and rotating, so that it is not restrictive at all.

Proof. We will argue by contradiction and assume that there exist sequences h_j^+ , r_j , A_j , b^j and a^j satisfying the hypothesis, such that

$$\sup_{K(r_j)} |h_j^+ + g_j| = jA_j^{\frac{1-\beta}{2}} r_j^{1+\beta},$$

where g_j satisfies (18). The proof is rather long so we will divide it into several claims. The first one will slightly modify $K(r_j)$ so that it respects the natural scaling of the Hamilton-Jacobi equation.

Claim 1: For

$$S_j(\sigma) = \sup_{B_1} \left(h_j^+ (\sqrt{\sigma} r_j x', r_j^{1-\beta} A_j^{-\frac{1-\beta}{2}} x_n) + g_j(r_j^{1-\beta} A_j^{-\frac{1-\beta}{2}} x_n) \right),$$

there exists σ_j such that $S_j(\sigma_j) = \sigma_j A_j^{\frac{1-\beta}{2}} r_j^{1+\beta}$ and $\sigma_j \rightarrow +\infty$.

Proof of Claim 1: By assumption $S_j(1) = jA_j^{\frac{1-\beta}{2}} r_j^{1+\beta} \gg A_j^{\frac{1-\beta}{2}} r_j^{1+\beta}$. Setting $\tau_j := 1/(A_j^{1-\beta} r_j^{2\beta})$ and using the one-sided estimates for h^+ (inequality (15)), we also obtain that

$$\begin{aligned} S_j(\tau_j) &\leq \sup_{B_{r_j^{1-\beta} A_j^{-(1-\beta)/2}}} |h_j^+| + \sup_{B_{r_j^{1-\beta} A_j^{-(1-\beta)/2}}} |g_j| \\ &\leq C A_j^{\frac{\beta^2-\beta}{2}} r_j^{1-\beta^2} + r_j^{(1-\beta)(1+\alpha)} A_j^{1-\frac{(1-\beta)(1+\alpha)}{2}} \\ &\leq C \left(\tau_j r_j^{1+\beta} A_j^{\frac{1-\beta}{2}} \right) A_j^{\frac{1-2\beta+\beta^2}{2}} r_j^{\beta-\beta^2} < \tau_j A_j^{(1-\beta)/2} r_j^{1+\beta}, \end{aligned}$$

provided that r_j is small enough.

By the continuity of S_j there is a $\sigma_j \in (1, \tau_j)$ such that $S_j(\sigma_j) = \sigma_j A_j^{\frac{1-\beta}{2}} r_j^{1+\beta}$.

Therefore we only need to show that $\sigma_j \rightarrow +\infty$. By definition we have

$$\sigma_j = \sup_{B_1} \frac{h_j^+ (\sqrt{\sigma_j} r_j x', r_j^{1-\beta} A_j^{-(1-\beta)/2} x_n) + g_j(r_j^{1-\beta} A_j^{-(1-\beta)/2} x_n)}{A_j^{(1-\beta)/2} r_j^{1+\beta}}$$

$$\geq \sup_{B_1} \frac{h_j^+(r_j x', r_j^{1-\beta} A_j^{-(1-\beta)/2} x_n) + g_j(r_j^{1-\beta} A_j^{-(1-\beta)/2} x_n)}{A_j^{(1-\beta)/2} r_j^{1+\beta}} = j,$$

which proves Claim 1.

Let us also remark that as

$$S_j(\sigma_j) = \sigma_j A_j^{\frac{1-\beta}{2}} r_j^{1+\beta},$$

we obtain from our one-sided estimates that

$$S_j(\sigma_j) \leq C \max \left(\sup_{B_{\sqrt{\sigma_j} r_j}} h_j^+ + \sup_{B_{\sqrt{\sigma_j} r_j}} |g_j|, \sup_{B_{A_j^{-(1-\beta)/2} r_j^{1-\beta}}} h_j^+ + \sup_{B_{A_j^{-(1-\beta)/2} r_j^{1-\beta}}} |g_j| \right).$$

Notice that, since $g'_j(t) = b(0', t)$,

$$\sup_{B_{A_j^{-(1-\beta)/2} r_j^{1-\beta}}} |g_j| \leq C \left(A_j^{\frac{1-\alpha+2\beta-(1+\alpha)\beta}{2}} r_j^{(\alpha-\beta)(1-\beta)} \right) A_j^{\frac{\beta^2-\beta}{2}} r_j^{1-\beta^2},$$

but the term in the parenthesis tends to zero, implying that

$$\sup_{B_{A_j^{-(1-\beta)/2} r_j^{1-\beta}}} |g_j| \leq A_j^{\frac{\beta^2-\beta}{2}} r_j^{1-\beta^2}.$$

Similarly,

$$\sup_{B_{\sqrt{\sigma_j} r_j}} |g_j| \leq A_j \sigma_j^{(1+\alpha)/2} r_j^{1+\alpha} \leq \left(\sigma_j^{\frac{\alpha-1}{2}} A_j^{\frac{1+\beta}{2}} r_j^{\alpha-\beta} \right) \sigma_j A_j^{\frac{1-\beta}{2}} r_j^{1+\beta} \leq S_j(\sigma_j).$$

Using these estimates on $|g_j|$ we arrive at

$$S_j(\sigma_j) \leq C \max \left(\sigma_j^{\frac{1+\beta}{2}} A_j^{\frac{1-\beta}{2}} r_j^{1+\beta}, A_j^{\frac{\beta^2-\beta}{2}} r_j^{1-\beta^2} \right).$$

Since $\sigma_j \rightarrow +\infty$ it follows that

$$\sigma_j A_j^{\frac{1-\beta}{2}} r_j^{1+\beta} \leq C A_j^{\frac{\beta^2-\beta}{2}} r_j^{1-\beta^2}.$$

We may—in order to simplify notation—assume that $\sigma_j = j$ and rewrite the above relation as

$$(19) \quad r_j \leq C A_j^{-\frac{1-\beta}{2\beta}} j^{-\frac{1}{\beta+\beta^2}}.$$

As mentioned before, Claim 1 gives an estimate that respects the scaling of h_j^+ : if we define h_j by

$$h_j(x) := \frac{h_j^+(\sqrt{j}r_jx', r_j^{1-\beta}A_j^{-\frac{1-\beta}{2}}x_n) + g_j(r_j^{1-\beta}A_j^{-\frac{1-\beta}{2}}x_n)}{jA_j^{\frac{1-\beta}{2}}r_j^{1+\beta}}$$

then $\sup_{B_1} |h_j| = 1$. Also

$$\nabla h_j(x) = \frac{\tilde{\nabla} h_j^+(y)}{\sqrt{j}A_j^{\frac{1-\beta}{2}}r_j^\beta} + \frac{\partial_n h_j^+(y) + g'_j(y_n)}{jA_j^{1-\beta}r_j^{2\beta}}e_n$$

where $y = (\sqrt{j}r_jx', r_j^{1-\beta}A_j^{-(1-\beta)/2}x_n)$. Alternatively we may write

$$\nabla h_j^+ = c_j \tilde{\nabla} h_j + (c_j^2 \partial_n h_j - g'_j)e_n,$$

where

$$c_j = \sqrt{j}A_j^{\frac{1-\beta}{2}}r_j^\beta.$$

Rewriting the Hamilton-Jacobi equation (1) in terms of our new function h_j and using assumption (a5), we may deduce that

$$(20) \quad \begin{aligned} & |\tilde{\nabla} h_j|^2 - 2\partial_n h_j = \\ & \frac{2b \cdot \tilde{\nabla} h_j}{c_j} + 2\partial_n h_j(b \cdot e_n + g'_j) - \frac{|b + e_n g'_j|^2}{c_j^2} - 2\frac{e_n \cdot b + g'_j}{c_j^2} - c_j^2(\partial_n h_j)^2 = \\ & T_1^j \cdot \tilde{\nabla} h_j + T_2^j + T_3^j + T_4^j - T_5^j. \end{aligned}$$

In order to use this equation we need to control the right-hand side. First we need to control the one-sided oscillation in the x' directions.

Claim 2: For every x_n and every p_j in the super-differential of h_j at

(x', x_n) we have for each \tilde{R} and any q_j in the super-differential of h_j^+ at $(0, r_j^{1-\beta}A_j^{-\frac{1-\beta}{2}})$ that

$$\sup_{B'_{\tilde{R}}} (h_j - h_j(0, x_n) - p_j \cdot x') \rightarrow 0 \text{ as } j \rightarrow \infty.$$

In particular, if $x_n = 0$ then $\text{osc}_{B'_{\tilde{R}}} (h_j(x', 0) - h_j(0) - p_j \cdot x') \rightarrow 0$ as $j \rightarrow \infty$.

Proof of Claim 2: The first statement is deduced from (a2) in the following way:

$$\sup_{B'_{\tilde{R}}} (h_j - h_j(0, x_n) - p_j \cdot x') \leq \sup_{B'_{\sqrt{j}r_j\tilde{R}}} \frac{h_j^+ - h_j^+(0, x_n) + q_j \cdot x'}{jA_j^{\frac{1-\beta}{2}}r_j^{1+\beta}} \leq C \frac{\tilde{R}^{1+\beta}}{j^{\frac{1-\beta}{2}}} \rightarrow 0$$

as $j \rightarrow \infty$. For the second part we use (17):

$$\inf_{B'_R} h_j(x', 0) \geq \inf_{B'_{\sqrt{j}r_j\bar{R}}} \frac{h_j^+(x', 0)}{jA_j^{\frac{1-\beta}{2}} r_j^{1+\beta}} \geq -C \frac{\bar{R}^{1+\beta}}{j^{\frac{1-\beta}{2}}} \rightarrow 0 \text{ as } j \rightarrow \infty.$$

In order to prove the proposition we want to show that $h_j \rightarrow h_0 \equiv 0$, which would clearly contradict uniform convergence and the fact that by definition $\sup_{B_1} |h_0| = \lim_j \sup_{B_1} |h_j| = 1$.

To show the convergence to 0 we need to control the right-hand side in equation (20). In particular, we are going to prove that $T_1^j, \dots, T_5^j \rightarrow 0$ as $j \rightarrow \infty$. We formulate this as a new claim.

Claim 3: We have $T_i^j \rightarrow 0$ locally uniformly as $j \rightarrow \infty$, for $i = 1, \dots, 5$, where T_i^j has been defined in equation (20).

Proof of Claim 3: Using that $|b(x)| \leq A|x|^\alpha$ and that $\frac{\alpha}{2+\alpha} \leq \beta \leq \alpha/2$, we see that

$$\sup_{B_1} |T_1^j| = \sup_{B_1} \frac{2|b|}{\sqrt{j}r_j^\beta A_j^{\frac{1-\beta}{2}}} \leq \frac{2A_j \left(A_j^{-\frac{(1-\beta)}{2}} r_j^{1-\beta} \right)^\alpha}{\sqrt{j} A_j^{\frac{1-\beta}{2}} r_j^\beta} = \frac{2A_j^{\frac{1-\alpha+\beta+\alpha\beta}{2}} r_j^{\alpha-\beta(1+\alpha)}}{\sqrt{j}} \rightarrow 0$$

as $j \rightarrow \infty$, which implies the claim for T_1^j .

We also have, by assumption, that $b(0, x_n) \cdot e_n + g'_j(x_n) = 0$, and thus

$$(21) \quad \sup_{B_1} |b(x) \cdot e_n - g'_j| \leq \sup_{|x_n| \leq A_j^{-\frac{1-\beta}{2}} r_j^{1-\beta}} \operatorname{osc}_{x' \in B_{\sqrt{j}r_j}} (b \cdot e_n) \leq A_j j^{\alpha/2} r_j^\alpha.$$

Using Lemma 6.6 we see that

$$(22) \quad \sup_{B_1} |\partial_n h_j| \leq \sup_{(x' / (\sqrt{j}r_j), A_j^{\frac{1-\beta}{2}} x_n / (r_j^{1-\beta})) \in B_1} \frac{|\partial_n h_j^+ + g'_j|}{jA_j^{1-\beta} r_j^{2\beta}} \leq \frac{cA_j^{-\frac{(1-\beta)^2}{2}} r_j^{-\beta-\beta^2}}{j}.$$

From (21), (22) and $\beta \leq \alpha/2 \leq 1/2$ we see that

$$\sup_{B_1} |T_2^j| \leq C \frac{A_j^{\frac{1+\beta^2}{2}} r_j^{\alpha-\beta-\beta^2}}{j^{1-\frac{\beta}{2}}} \rightarrow 0$$

as $j \rightarrow \infty$. Next, we estimate in B_1

$$(23) \quad |T_3^j| \leq \frac{2|b|^2}{jA_j^{1-\beta} r_j^{2\beta}} \leq \frac{2A_j^{2-(1-\beta)\alpha} r_j^{2\alpha(1-\beta)}}{jA_j^{1-\beta} r_j^{2\beta}} = \frac{2A_j^{1+\beta-(1-\beta)\alpha} r_j^{2(\alpha-\beta(1+\alpha))}}{j} \rightarrow 0$$

as $j \rightarrow \infty$. The term T_4^j satisfies

$$|T_4^j| \leq \frac{|b \cdot e_n + g'_j|}{jA_j^{1-\beta}r_j^{2\beta}} \leq \sup_{|x_n| \leq r_j^{1-\beta}} \sup_{x' \in B_{\sqrt{j}r_j}} \text{osc} \left(\frac{b \cdot e_n}{jA_j^{1-\beta}r_j^{2\beta}} \right) \leq \frac{A_j^\beta r_j^{\alpha-2\beta}}{j^{1-\alpha/2}} \rightarrow 0$$

as $j \rightarrow \infty$.

So far we have shown that $T_1^j, \dots, T_4^j \rightarrow 0$ as $j \rightarrow \infty$.

Next, we derive estimates for T_5^j . First we use (19) to arrive at

$$\{x; (x'/(\sqrt{j}r_j), A_j^{(1-\beta)/2}x_n/r_j^{1-\beta}) \in B_1\} \subset B_{A_j^{-(1-\beta)/2}r_j^{1-\beta}}.$$

Together with Lemma 6.6 this implies that

$$|\tilde{\nabla}h_j^+| \leq CA_j^{\frac{1-\beta}{2}} A_j^{-\frac{1-\beta}{2}\beta} r_j^{(1-\beta)\beta} \quad \text{in } \{x; (x'/(\sqrt{j}r_j), A_j^{\frac{1-\beta}{2}}x_n/r_j^{1-\beta}) \in B_1\},$$

that is

$$(24) \quad |\tilde{\nabla}h_j| \leq C \frac{A_j^{-\frac{1-\beta}{2}\beta} r_j^{-\beta^2}}{\sqrt{j}},$$

Now, since h_j is a Lipschitz solution to the Hamilton-Jacobi equation, the super-differential consists almost everywhere of a unique element p , and almost everywhere, $\nabla h_j = p$ solves equation (20). That is, at almost every point,

$$p_n^2 - \frac{2}{jA_j^{1-\beta}r_j^{2\beta}}p_n + \frac{\delta}{jA_j^{1-\beta}r_j^{2\beta}} = 0,$$

where

$$\begin{aligned} \delta = |\tilde{\nabla}h_j|^2 - \frac{2b}{\sqrt{j}A_j^{\frac{1-\beta}{2}}r_j^\beta} \cdot \tilde{\nabla}h_j - 2\partial_n h_j((b(x) \cdot e_n + g'_j) \\ + \frac{|b + e_n g'_j|^2}{jA_j^{1-\beta}r_j^{2\beta}} + 2\frac{e_n \cdot b + g'_j}{jA_j^{1-\beta}r_j^{2\beta}}. \end{aligned}$$

This means that p_n can only take the two values (depending on δ),

$$(25) \quad p_n = \begin{cases} \frac{1}{jA_j^{1-\beta}r_j^{2\beta}} + \sqrt{\frac{1}{j^2A_j^{2-2\beta}r_j^{4\beta}} - \frac{\delta}{jA_j^{1-\beta}r_j^{2\beta}}} \\ \frac{1}{jA_j^{1-\beta}r_j^{2\beta}} - \sqrt{\frac{1}{j^2A_j^{2-2\beta}r_j^{4\beta}} - \frac{\delta}{jA_j^{1-\beta}r_j^{2\beta}}} \end{cases}$$

By (22) as well as the above estimates on $|\tilde{\nabla}h_j|$ (c.f. equation (24)) we see that

$$(26) \quad |\delta| \leq C \left[\frac{A_j^{-(1-\beta)\beta} r_j^{-2\beta^2}}{j} + \frac{A_j^{\frac{1-\alpha+\beta+\beta^2}{2}} r_j^{\alpha-\beta(1+\alpha+\beta)}}{j} \right. \\ \left. + \frac{A_j^{\frac{1+2\beta-\beta^2}{2}} r_j^{\alpha-\beta-\beta^2}}{j^{1-\frac{\alpha}{2}}} + \frac{A_j^{1+\beta-(1-\beta)\alpha} r_j^{2(\alpha-\beta(1+\alpha))}}{j} + \frac{A_j^\beta r_j^{\alpha-2\beta}}{j^{1-\frac{\alpha}{2}}} \right].$$

Next, using (19) and (23) we see that when j is large and r_j is small,

$$(27) \quad \frac{A_j^{\frac{1+2\beta-\beta^2}{2}} r_j^{\alpha-\beta-\beta^2}}{j^{1-\frac{\alpha}{2}}} \leq \left(A_j^{\frac{1-\beta^2}{2}} r_j^{\beta-\beta^2} \right) \frac{A_j^\beta r_j^{\alpha-2\beta}}{j^{1-\frac{\alpha}{2}}} \ll \frac{A_j^\beta r_j^{\alpha-2\beta}}{j^{1-\frac{\alpha}{2}}}.$$

Similarly, using (19) which implies that

$$r_j^{\beta-\alpha\beta-\beta^2} \leq j^{\frac{\alpha+\beta-1}{1+\beta}} A_j^{-\frac{(1-\beta)(1-\alpha-\beta)}{2}},$$

we may conclude that

$$(28) \quad \frac{A_j^{\frac{1-\alpha+\beta+\beta^2}{2}} r_j^{\alpha-\beta(1+\alpha)-\beta^2}}{j} = \left(\frac{A_j^{\frac{1-\alpha+\beta+\beta^2}{2}-2\beta}}{j^{\alpha/2}} r_j^{\beta-\alpha\beta-\beta^2} \right) \frac{A_j^\beta r_j^{\alpha-2\beta}}{j^{1-\alpha/2}} \\ \leq j^{\frac{\alpha+\beta-1}{1+\beta}-\frac{\alpha}{2}} \frac{A_j^\beta r_j^{\alpha-2\beta}}{j^{1-\alpha/2}},$$

Noticing that the term $j^{\frac{\alpha+\beta-1}{1+\beta}-\frac{\alpha}{2}}$ in (28) goes to zero as $j \rightarrow \infty$, we deduce that

$$(29) \quad \frac{A_j^{\frac{1-\alpha+\beta+2\alpha\beta+\beta^2}{2}} r_j^{\alpha-\beta(1+\alpha)-\beta^2}}{j} \ll \frac{A_j^\beta r_j^{\alpha-2\beta}}{j^{1-\alpha/2}} \text{ as } j \rightarrow \infty.$$

Moreover, as $2\beta \leq \alpha \leq 1$, we have

$$(30) \quad \frac{A_j^{1+\beta-(1-\beta)\alpha} r_j^{2(\alpha-\beta(1+\alpha))}}{j} \ll \frac{A_j^{-(1-\beta)\beta} r_j^{-2\beta^2}}{j}$$

when r_j is small. Using (26)-(30), we estimate

$$|\delta| \leq C \left[\frac{A_j^{-(1-\beta)\beta} r_j^{-2\beta^2}}{j} + \frac{A_j^\beta r_j^{\alpha-2\beta}}{j^{1-\frac{\alpha}{2}}} \right].$$

Using (19) again along with $\alpha/(2 + \alpha) \leq \beta \leq \alpha/2 \leq 1/2$, we see that

$$(31) \quad \frac{A_j^{-(1-\beta)\beta} r_j^{-2\beta^2}}{j} = \left(\frac{1}{j^{\frac{2\beta-\beta^2}{\beta+\beta^2}}} \right) \frac{A_j^{-(1-\beta)} r_j^{-2\beta}}{j} \ll \frac{A_j^{-(1-\beta)} r_j^{-2\beta}}{j}$$

and that

$$(32) \quad \frac{A_j^\beta r_j^{\alpha-2\beta}}{j^{1-\frac{\alpha}{2}}} \leq \left(\frac{A_j^{\frac{2\beta-\alpha+\alpha\beta}{2\beta}}}{j^{\frac{-\alpha(\beta+\beta^2)+2\alpha}{2(\beta+\beta^2)}}} \right) \frac{A_j^{-(1-\beta)} r_j^{-2\beta}}{j} \ll \frac{A_j^{-(1-\beta)} r_j^{-2\beta}}{j}$$

when j is large. From (25), (31) and (32) we see that we can make a Taylor expansion in equation (25) and deduce that p_n is of order

$$(33) \quad |p_n| \approx \begin{cases} \frac{2}{j A_j^{1-\beta} r_j^{2\beta}} \\ \frac{|\delta|}{2} \leq \frac{C A_j^{-(1-\beta)\beta} r_j^{-2\beta^2}}{j} + \frac{C A_j^\beta r_j^{\alpha-2\beta}}{j^{1-\frac{\alpha}{2}}}. \end{cases}$$

Now remember that from (22), we have the additional information that

$$|\partial_n h_j| \leq \frac{C A_j^{-\frac{1-2\beta+\beta^2}{2}} r_j^{-\beta-\beta^2}}{j}.$$

As $\beta \leq 1/2$, we conclude that for small r_j , the first line in (33) and not the second must hold. Thus

$$|p_n| = \frac{|\delta|}{2} + \text{lower order terms.}$$

It follows that almost everywhere,

$$T_5 = j A_j^{1-\beta} r_j^{2\beta} (p_n)^2 \leq C \left(\frac{A_j^{1-3\beta+2\beta^2} r_j^{2\beta-4\beta^2}}{j} + \frac{A_j^{1+\beta} r_j^{2(\alpha-\beta)}}{j^{1-\alpha}} \right).$$

Since $2\beta \leq \alpha \leq 1$, the right-hand goes to zero as $j \rightarrow \infty$, and the claim follows.

The proof of the proposition is now easy to finish. We know from Claim 3 and Lemma 6.4 that $h_j \rightarrow h_0$ locally uniformly in \mathbb{R}^n , where h_0 is a viscosity solution to

$$(34) \quad |\nabla' h_0|^2 - 2\partial_n h_0 = 0.$$

Moreover $h_j \rightarrow 0$ on $\{x_n = 0\}$ by Claim 2. Using uniqueness for parabolic Hamilton-Jacobi equations ([?]) we deduce that $h_0 = 0$ for $x_n \leq 0$ and $h_0 \geq 0$ in $\{x_n \geq 0\}$. By Claim 2 we also know that h_0 is concave in the x' directions, which together with the bound from

below implies that h_0 is constant for each $x_n \geq 0$: if $h_0(x', x_n)$ is not constant for some $x_n = t > 0$, then $h_0(x', t) \leq h_0(0, t) + p \cdot x'$ for some non-zero $p \in \mathbb{R}^{n-1}$. It follows that there is a point x'_0 such that $h_0(x'_0, t) < 0$. But from (34) it follows that $\partial_n h_0 \geq 0$, a contradiction. Therefore $h_0(x) = h_0(x_n)$. Using (34) again we see that $\partial_n h_0 = 0$ in $\{x_n \geq 0\}$, and thus $h_0 = 0$. This contradicts $\sup_{B_1} |h_0| = 1$, and the Proposition follows. \square

4. IMPROVED REGULARITY IN DEGENERATE DIRECTIONS

In this section we improve the regularity in the directions orthogonal to the non-degenerate direction. The idea is to use the scaling from Proposition 3.2 to get a parabolic equation and deduce better regularity in the x' directions from the regularity theory for parabolic Hamilton-Jacobi equations.

Proposition 4.1. *Let h^+ be a viscosity solution to $|\nabla h^+ - a|^2 = 1$ in B_R and assume that h^+ satisfies the one-sided $C^{1,\beta}$ estimate from above*

$$(35) \quad \sup_{B_r(0)} \left(h^+(x) - h^+(0) - p \cdot (x - 0) \right) \leq CA^{\frac{1-\beta}{2}} r^{1+\beta}.$$

Furthermore let

$$(36) \quad h^+(x', x_n) \leq h^+(0, x_n) + p \cdot x' + CA^{\frac{1-\beta}{2}} |x'|^{1+\beta}$$

for each fixed x_n , and assume that h^+ satisfies the one-sided $C^{1,\beta}$ -estimate from below at the origin

$$(37) \quad h^+(x) \geq -CA^{\frac{1-\beta}{2}} |x|^{1+\beta}.$$

Also assume that

$$(38) \quad h^+(x', 0) \geq h^+(0) + p \cdot x' - CA^{\frac{1-\beta}{2}} |x'|^{1+\beta}.$$

In the just mentioned estimates we assume moreover that the constant $A \leq \tilde{C}$ (where \tilde{C} is a fixed constant) controls the seminorm of $a(x)$ by $[a]_{C^\alpha} \leq A$. Finally we assume that $a(x) = e_n + b(x)$ where $b(0) = 0$ and that $\alpha/(2 + \alpha) \leq \beta \leq \alpha/2$.

Then for every $\delta \leq \min(A^{1/2} R^{\frac{1}{1-\beta}}, R)$ and every $x_n \in (0, A^{-\frac{1-\beta}{2}} \delta^{1-\beta})$ we have for some $p \in \mathbb{R}^n$

$$\sup_{B'_\delta(0, x_n)} \left(h^+(x) - h^+(0, x_n) - p \cdot x' \right) \leq Q$$

where

$$Q = \begin{cases} CA^{\frac{1}{2} - \frac{\alpha\beta}{2(2+\alpha-2\beta)}} \delta^{1+\frac{\alpha}{2+\alpha-2\beta}} & \text{if } \delta \leq \min\left(A^{\frac{6\beta+\alpha\beta-2-\alpha}{2(4\beta-\alpha)}}, A^{-\frac{\beta}{\alpha-2\beta}}, A^{\frac{\beta}{2}} R^{\frac{2+\alpha-2\beta}{2}}\right), \\ CA^{\frac{1-\beta}{2}} R^{\beta-1} \delta^2 & \text{if } A^{\frac{\beta}{2}} R^{\frac{2+\alpha-2\beta}{2}} \leq \delta \leq A^{\frac{6\beta+\alpha\beta-2-\alpha}{2(4\beta-\alpha)}}, \\ CA^{\frac{1+\beta}{2}} \delta^{1+\alpha-\beta} & \text{if } A^{-\frac{\beta}{\alpha-2\beta}} \leq \delta \leq A^{\frac{6\beta+\alpha\beta-2-\alpha}{2(4\beta-\alpha)}}, \\ CA^{\frac{1-\beta}{1+\beta}} \delta^{1+\frac{2\beta}{1+\beta}} & \text{if } A^{\frac{6\beta+\alpha\beta-2-\alpha}{2(4\beta-\alpha)}} \leq \delta \leq \min\left(A^{-\frac{1-\beta}{2\beta}}, A^{\frac{1-\beta}{2}} R^{1+\beta}\right), \\ CA^{\frac{3(1-\beta)}{2}} \delta^{1+3\beta} & \text{if } \min\left[\max\left(A^{-\frac{1-\beta}{2\beta}}, A^{\frac{6\beta+\alpha\beta-2-\alpha}{2(4\beta-\alpha)}}\right), \right. \\ & \left. \max\left(A^{\frac{1-\beta}{2}} R^{1+\beta}, A^{\frac{6\beta+\alpha\beta-2-\alpha}{2(4\beta-\alpha)}}\right)\right] \leq \delta \end{cases}$$

if $R = \min(A^{1/2} R^{\frac{1}{1-\beta}}, R)$, and

$$Q = \begin{cases} CA^{\frac{1}{2} - \frac{\alpha\beta}{2(2+\alpha-2\beta)}} \delta^{1+\frac{\alpha}{2+\alpha-2\beta}} & \text{if } \delta \leq \min\left(A^{\frac{6\beta+\alpha\beta-2-\alpha}{2(4\beta-\alpha)}}, A^{-\frac{\beta}{\alpha-2\beta}}, A^{\frac{2+\alpha}{4}} R^{\frac{2+\alpha-2\beta}{2(1-\beta)}}\right), \\ CR^{-1} \delta^2 & \text{if } A^{\frac{2+\alpha}{4}} R^{\frac{2+\alpha-2\beta}{2(1-\beta)}} \leq \delta \leq A^{\frac{6\beta+\alpha\beta-2-\alpha}{2(4\beta-\alpha)}}, \\ CA^{\frac{1+\beta}{2}} \delta^{1+\alpha-\beta} & \text{if } A^{-\frac{\beta}{\alpha-2\beta}} \leq \delta \leq A^{\frac{6\beta+\alpha\beta-2-\alpha}{2(4\beta-\alpha)}}, \\ CA^{\frac{1-\beta}{1+\beta}} \delta^{1+\frac{2\beta}{1+\beta}} & \text{if } A^{\frac{6\beta+\alpha\beta-2-\alpha}{2(4\beta-\alpha)}} \leq \delta \leq AR^{\frac{1+\beta}{1-\beta}}, \\ CR^{-1} \delta^2 & \text{if } \max\left(AR^{\frac{1+\beta}{1-\beta}}, A^{\frac{6\beta+\alpha\beta-2-\alpha}{2(4\beta-\alpha)}}\right) \leq \delta \end{cases}$$

if $A^{1/2} R^{\frac{1}{1-\beta}} = \min(A^{1/2} R^{\frac{1}{1-\beta}}, R)$.

Proof: From Proposition 3.2 we obtain that

$$h(x) = \frac{h^+(rx', A^{-(1-\beta)/2} r^{1-\beta} x_n) + g(A^{-(1-\beta)/2} r^{1-\beta} x_n)}{A^{(1-\beta)/2} r^{1+\beta}}$$

is bounded from above. Moreover we know from the proof of Proposition 3.2 that h solves (with slightly reordered terms as compared to equation (20))

$$\begin{aligned} (39) \quad & \left| \tilde{\nabla} h - \frac{b'(0, A^{-(1-\beta)/2} r^{1-\beta} x_n)}{A^{(1-\beta)/2} r^\beta} \right|^2 - 2\partial_n h \\ &= -A^{1-\beta} r^{2\beta} |\partial_n h|^2 + 2 \frac{b(rx', A^{-(1-\beta)/2} r^{1-\beta} x_n) - b(0, A^{-(1-\beta)/2} r^{1-\beta} x_n)}{A^{(1-\beta)/2} r^\beta} \cdot \tilde{\nabla} h \\ & - 2 \frac{b(rx', A^{-(1-\beta)/2} r^{1-\beta} x_n) \cdot e_n + g'}{A^{1-\beta} r^{2\beta}} + 2(b(rx', A^{-(1-\beta)/2} r^{1-\beta} x_n) \cdot e_n + g') \partial_n h \\ & - \frac{|b(rx', A^{-(1-\beta)/2} r^{1-\beta} x_n) + g' e_n|^2}{A^{1-\beta} r^{2\beta}} - \frac{|b'(0, A^{-(1-\beta)/2} r^{1-\beta} x_n)|^2}{A^{1-\beta} r^{2\beta}} \\ &= -I - II - III - IV - V - VI, \end{aligned}$$

where we have used the notation $b' = (b_1, b_2, \dots, b_{n-1}, 0)$ in the first line of the equation.

Let us first show that the right-hand is bounded: Using exactly the same argument as in the estimate of T_5 in Claim 3 of Proposition 3.2 (consider the same argument with $j := 1$), we see that

$$(40) \quad \sup_{B_1} |I| \leq C(A^{1-3\beta+2\beta^2} r^{2\beta(1-2\beta)} + A^{1+\beta} r^{2(\alpha-\beta)}) \leq C.$$

Next,

$$(41) \quad \begin{aligned} \sup_{B_1} |II| &\leq \sup_{B_1} \frac{|b(rx', A^{-(1-\beta)/2} r^{1-\beta} x_n) - b(0, A^{-(1-\beta)/2} r^{1-\beta} x_n)|}{A^{(1-\beta)/2} r^\beta} |\tilde{\nabla} h| \\ &\leq 2A^{\frac{1+\beta}{2}} r^{\alpha-\beta} \sup_{B_1} |\tilde{\nabla} h(x', x_n)|. \end{aligned}$$

Similarly we may estimate

$$(42) \quad \sup_{B_1} |III| \leq \sup_{|x_n| \leq A^{\frac{-1+\beta}{2}} r^{1-\beta}} \operatorname{osc}_{B'_r} \left(\frac{b \cdot e_n}{A^{1-\beta} r^{2\beta}} \right) \leq 4A^\beta r^{\alpha-2\beta} \leq C,$$

$$(43) \quad \sup_{B_1} |IV| \leq 2A^{(1+\beta)/2} r^{\alpha-\beta} \operatorname{osc}_{B_1} \sqrt{I} \leq C,$$

$$(44) \quad \sup_{B_1} |V| \leq CA^{(1-\alpha)+(1+\alpha)\beta} r^{2\alpha-2\beta(1+\alpha)} \leq C,$$

$$(45) \quad \sup_{B_1} |VI| \leq CA^{(1-\beta)(1-\alpha)} r^{2(\alpha-\beta-\alpha\beta)} \leq C.$$

From (40), (41), (42), (43), (44) and (45) as well as the parabolic comparison principle Lemma 6.4, we obtain that $|\nabla h| \leq C$ in $Q_1 = \{x; |x'| \leq 1, 0 < x_n < 1\}$.

Next we are going to estimate the oscillation of each term of the right-hand side in equation (39) in the x' variable: First we notice that the oscillation of VI in the x' directions is identically zero.

Next, using the gradient bound of h , we obtain that for every $0 < x_n < 1$,

$$(46) \quad \operatorname{osc}_{B'_1} I \leq CA^{1-\beta} r^{2\beta},$$

$$(47) \quad \operatorname{osc}_{B'_1} II \leq 2A^{(1+\beta)/2} r^{\alpha-\beta},$$

$$(48) \quad \operatorname{osc}_{B'_1} III \leq 4A^\beta r^{\alpha-2\beta},$$

$$(49) \quad \operatorname{osc}_{B'_1} IV \leq CA r^\alpha,$$

$$(50) \quad \operatorname{osc}_{B'_1} V \leq 2A^{1+\beta} r^{2(\alpha-\beta)}.$$

So the oscillation in x' of the right-hand side in equation (39) is estimated by $q = C \max(A^{1-\beta} r^{2\beta}, A^\beta r^{\alpha-2\beta})$, where we have used that in view of $2\beta \leq \alpha \leq 1$, $\text{osc}_{B'_1} I$ or $\text{osc}_{B'_1} III$ is dominating when r is small.

Now let w be the solution to

$$\begin{aligned} |\tilde{\nabla} w|^2 - 2w_n &= q + f(x_n) && \text{in } Q_1, \\ w &= h && \text{on } \partial Q_1 \setminus \{x_n = 0\}, \end{aligned}$$

where $f(x_n)$ is chosen so that the right-hand side of the above equation equals the right-hand side of (39) at $|x'| = 0$. By the parabolic comparison principle Lemma 6.4 and the one-sided estimate Lemma 6.2 for viscosity solutions we have for p in the super-differential of w at x_0 that

$$\begin{aligned} h(x) &\leq w(x) \leq w(x_0) + p \cdot (x' - x'_0) + C|x' - x'_0|^2 \\ &\leq h(x_0) + p \cdot (x' - x'_0) + C|x' - x'_0|^2 + Cq. \end{aligned}$$

Rescaling back to h^+ we see that this implies that (for some p in the super-differential of h^+)

$$h^+(x', x_n) \leq h^+(0, x_n) + p \cdot x' + C(A^{(1-\beta)/2} r^{1+\beta} q + A^{(1-\beta)/2} r^{\beta-1} |x'|^2)$$

for $0 < x_n < A^{-\frac{1-\beta}{2}} r^{1-\beta}$ and $|x'| \leq r$.

Using this estimate in an optimal way will yield the proposition. Rearranging terms and taking the supremum in B'_δ on both sides leads to

$$(51) \quad \sup_{B'_\delta} (h^+(x', x_n) - h^+(0, x_n) - p \cdot x') \leq C(A^{(1-\beta)/2} r^{1+\beta} q + A^{(1-\beta)/2} r^{\beta-1} \delta^2)$$

provided that $\delta \in (0, r)$ and $0 < x_n < A^{-\frac{1-\beta}{2}} \delta^{1-\beta}$.

For fixed R and fixed $\delta \in (0, R)$ we want to find an r optimizing this estimate. Since we have a constraint that $r \leq \min(A^{1/2} R^{\frac{1}{1-\beta}}, R)$ it is natural to divide the proof into two cases.

Case 1, $R = \inf(A^{1/2} R^{\frac{1}{1-\beta}}, R)$: That is, when

$$R \geq A^{-\frac{1-\beta}{2\beta}}.$$

When $q = CA^\beta r^{\alpha-2\beta}$, or equivalently when

$$(52) \quad r \leq A^{-\frac{1-2\beta}{4\beta-\alpha}},$$

then we would want to choose

$$(53) \quad r = A^{-\frac{\beta}{2+\alpha-2\beta}} \delta^{\frac{2}{2+\alpha-2\beta}}.$$

From (52) we infer then that

$$(54) \quad A^{-\frac{\beta}{2+\alpha-2\beta}} \delta^{\frac{2}{2+\alpha-2\beta}} \leq A^{-\frac{1-2\beta}{4\beta-\alpha}},$$

or equivalently that

$$(55) \quad \delta \leq A^{\frac{6\beta+\alpha\beta-2-\alpha}{2(4\beta-\alpha)}}.$$

Moreover we need $r \geq \delta$ and $r \leq R$ in order to use (51), that is

$$(56) \quad \delta \leq A^{-\frac{\beta}{\alpha-2\beta}}$$

and

$$(57) \quad \delta \leq A^{\frac{\beta}{2}} R^{\frac{2+\alpha-2\beta}{2}}.$$

When (52), (56) and (57) hold, then we can choose r according to (53) and deduce that

$$\sup_{B'_\delta} (h^+(x', x_n) - h^+(0, x_n) - p \cdot x') \leq C A^{\frac{1}{2} - \frac{\alpha\beta}{2(2+\alpha-2\beta)}} \delta^{1 + \frac{\alpha}{2+\alpha-2\beta}}.$$

It still remains to consider the cases when at least one of (52), (56) or (57) does not hold.

If (52) holds but not (57) then we choose $r = R$ and we deduce that—using $q = A^\beta r^{\alpha-2\beta}$ which is equivalent to (54)—

$$(58) \quad \sup_{B'_\delta} (h^+(x', x_n) - h^+(0, x_n) - p \cdot x') \leq C \left(A^{\frac{1-\beta}{2}} R^{1+\beta} q + A^{\frac{1-\beta}{2}} R^{\beta-1} \delta^2 \right) \\ = C \left(A^{\frac{1+\beta}{2}} R^{1+\alpha-\beta} + A^{\frac{1-\beta}{2}} R^{\beta-1} \delta^2 \right).$$

In order to simplify (58) we use that $\delta > A^{\frac{\beta}{2}} R^{\frac{2+\alpha-2\beta}{2}}$ and thus

$$A^{\frac{1-\beta}{2}} R^{\beta-1} \delta^2 \geq A^{\frac{1+\beta}{2}} R^{1+\alpha-\beta}.$$

This way we may simplify (58) to

$$(59) \quad \sup_{B'_\delta} (h^+(x', x_n) - h^+(0, x_n) - p \cdot x') \leq C A^{\frac{1-\beta}{2}} R^{\beta-1} \delta^2.$$

If (52) holds but (56) does not, then we choose $r = \delta$ and deduce from (51) that

$$(60) \quad \sup_{B'_\delta} (h^+(x', x_n) - h^+(0, x_n) - p \cdot x') \leq C \left(A^{\frac{1+\beta}{2}} \delta^{1+\alpha-\beta} + A^{\frac{1-\beta}{2}} \delta^{1+\beta} \right).$$

The information that (56) does not hold implies

$$\delta \geq A^{-\frac{\beta}{\alpha-\beta}},$$

so we may estimate

$$A^{\frac{1+\beta}{2}} \delta^{1+\alpha-\beta} = (A^\beta \delta^{\alpha-2\beta}) A^{\frac{1-\beta}{2}} \delta^{1+\beta} \geq A^{\frac{1-\beta}{2}} \delta^{1+\beta}.$$

We can thus simplify (60) to

$$(61) \quad \sup_{B'_\delta} (h^+(x', x_n) - h^+(0, x_n) - p \cdot x') \leq CA^{\frac{1+\beta}{2}} \delta^{1+\alpha-\beta}.$$

In case (52) does not hold—that is when $q = CA^{1-\beta} r^{2\beta}$ —our first choice of r is $r = A^{-\frac{1-\beta}{2(1+\beta)}} \delta^{\frac{1}{1+\beta}}$; notice that by the condition $r \leq R$, this is only possible for

$$(62) \quad \delta \leq A^{\frac{1-\beta}{2}} R^{1+\beta}.$$

Moreover we must have that $r \geq \delta$ which in this case becomes

$$(63) \quad \delta \leq A^{-\frac{1-\beta}{2\beta}}.$$

So in case (52) does not hold but (62) and (63) do we obtain from (51) that

$$(64) \quad \sup_{B'_\delta} (h^+(x', x_n) - h^+(0, x_n) - p \cdot x') \leq CA^{\frac{1-\beta}{1+\beta}} \delta^{1+\frac{2\beta}{1+\beta}}.$$

If neither (52) nor (63) hold then we chose $r = 2\delta$ and deduce that

$$(65) \quad \sup_{B'_\delta} (h^+(x', x_n) - h^+(0, x_n) - p \cdot x') \leq C \left(A^{\frac{3(1-\beta)}{2}} \delta^{1+3\beta} + A^{\frac{1-\beta}{2}} \delta^{1+\beta} \right).$$

In order to simplify (65) we notice that as (63) does not hold, we have $\delta \geq A^{-\frac{1-\beta}{2\beta}}$ and thus

$$A^{\frac{3(1-\beta)}{2}} \delta^{1+3\beta} \geq A^{\frac{1-\beta}{2}} \delta^{1+\beta}.$$

Therefore we may write (65) as

$$(66) \quad \sup_{B'_\delta} (h^+(x', x_n) - h^+(0, x_n) - p \cdot x') \leq CA^{\frac{3(1-\beta)}{2}} \delta^{1+3\beta}.$$

Finally, if neither (52) nor (62) hold, we use $r = R$ in equation (51) and $\delta \geq A^{\frac{1-\beta}{2}} R^{1+\beta}$ to obtain

$$\sup_{B'_\delta} (h^+ - h^+(0, x_n) - p \cdot x') \leq C \left(A^{\frac{1+\beta}{2}} R^{1+\alpha-\beta} + A^{\frac{1-\beta}{2}} R^{\beta-1} \delta^2 \right) \leq CA^{\frac{3(1-\beta)}{2}} \delta^{1+3\beta}.$$

This is the final estimate on Q in Case 1.

Case 2, $A^{1/2} R^{\frac{1}{1-\beta}} = \inf(A^{1/2} R^{\frac{1}{1-\beta}}, R)$: The argument is similar as in Case 1. In this case we have $R \leq A^{-\frac{1-\beta}{2\beta}}$. Again, if $q = CA^\beta r^{\alpha-2\beta}$ then

the best balance in (51) is obtained by the choice $r = A^{-\frac{\beta}{2+\alpha-2\beta}} \delta^{\frac{2}{2+\alpha-2\beta}}$.

But when $q = CA^\beta r^{\alpha-2\beta}$, then $r \leq A^{-\frac{1-2\beta}{4\beta-\alpha}}$ which is equivalent to

$$(67) \quad \delta \leq A^{\frac{6\beta+\alpha\beta-2-\alpha}{2(4\beta-\alpha)}}$$

when $r = A^{-\frac{\beta}{2+\alpha-2\beta}} \delta^{\frac{2}{2+\alpha-2\beta}}$. In order to use (51) we need $r \geq \delta$ which with our choice of r is equivalent to

$$(68) \quad \delta \leq A^{-\frac{\beta}{\alpha-2\beta}}.$$

Moreover, as $r \leq A^{\frac{1}{2}} R^{\frac{1}{1-\beta}}$, we must have

$$(69) \quad \delta \leq A^{\frac{2+\alpha}{4}} R^{\frac{2+\alpha-2\beta}{2(1-\beta)}}$$

for our choice of r .

If (67), (68) and (69) hold, then (51) implies that

$$(70) \quad \sup_{B'_\delta} (h^+ - h^+(0, x_n) - p \cdot x') \leq CA^{\frac{1}{2} - \frac{\alpha\beta}{2(2+\alpha-2\beta)}} \delta^{1 + \frac{\alpha}{2+\alpha-2\beta}}.$$

As before we also need to investigate what happens if at least One of the conditions $q = CA^\beta r^{\alpha-2\beta}$, (68) or (69) does not hold.

If (67) holds but (69) does not, then we choose $r = A^{\frac{1}{2}} R^{\frac{1}{1-\beta}}$ and deduce from (51) that

$$(71) \quad \sup_{B'_\delta} (h^+ - h^+(0, x_n) - p \cdot x') \leq C \left(A^{\frac{1+\beta}{2}} r^{1+\alpha-\beta} + A^{\frac{1-\beta}{2}} R^{\beta-1} \delta^2 \right) \\ = C \left(A^{\frac{2+\alpha}{2}} R^{\frac{1+\alpha-\beta}{1-\beta}} + R^{-1} \delta^2 \right).$$

On the other hand, the information that (69) does not hold implies that $R^{-1} \delta^2 \geq A^{\frac{2+\alpha}{2}} R^{\frac{1+\alpha-\beta}{1-\beta}}$. We may therefore simplify (71) to

$$(72) \quad \sup_{B'_\delta} (h^+ - h^+(0, x_n) - p \cdot x') \leq CR^{-1} \delta^2.$$

Next, if (67) holds but (68) does not, we choose $r = 2\delta$ and deduce that —using that (68) does not hold—

$$(73) \quad \sup_{B'_\delta} (h^+ - h^+(0, x_n) - p \cdot x') \leq C \left(A^{\frac{1+\beta}{2}} \delta^{1+\alpha-\beta} + A^{\frac{1-\beta}{2}} \delta^{1+\beta} \right) \\ \leq CA^{\frac{1+\beta}{2}} \delta^{1+\alpha-\beta}.$$

If (67) does not hold, then $q = CA^{1-\beta} r^{2\beta}$ and the optimal choice of r in (51) is given by

$$(74) \quad r = A^{-\frac{1-\beta}{2(1+\beta)}} \delta^{\frac{1}{1+\beta}}.$$

In order to satisfy $\delta \leq r$, we require furthermore that

$$(75) \quad \delta \leq A^{-\frac{1-\beta}{2\beta}}.$$

And in order to satisfy $r \leq A^{\frac{1}{2}} R^{\frac{1}{1-\beta}}$ we require that

$$(76) \quad \delta \leq AR^{\frac{1+\beta}{1-\beta}}.$$

From (75), (76), (74) as well as the information that (67) does not hold we infer that

$$(77) \quad \sup_{B'_\delta} (h^+ - h^+(0, x_n) - p \cdot x') \leq CA^{\frac{1-\beta}{1+\beta}} \delta^{1+\frac{2\beta}{1+\beta}}.$$

Notice that (75) is always satisfied in **Case 2** as $\delta < R \leq A^{-\frac{1-\beta}{2\beta}}$.

We therefore only need to check what happens if neither (76) nor (67) hold in order to finish the proof. In this case we choose $r = A^{\frac{1}{2}} R^{\frac{1}{1-\beta}}$. With this choice, (51) implies that —using the fact that (76) does not hold in the last of the following inequalities—

$$(78) \quad \begin{aligned} \sup_{B'_\delta} (h^+ - h^+(0, x_n) - p \cdot x') &\leq C \left(A^{\frac{1(1-\beta)}{2}} r^{1+3\beta} + A^{\frac{1-\beta}{2}} r^{\beta-1} \delta^2 \right) \\ &= C \left(A^2 R^{\frac{1+3\beta}{1-\beta}} + R^{-1} \delta^2 \right) = CR^{-1} \delta^2. \end{aligned}$$

□

5. REGULARITY FOR OBSTACLE PROBLEMS (PROOF OF MAIN THEOREM)

5.1. Heuristic arguments. In this section we combine the results from our previous sections and prove our main theorem, i.e. optimal regularity of the solution to the obstacle problem with Hamilton-Jacobi obstacle.

The proof is again somewhat involved, so before we start let us describe the idea.

Our goal is to prove optimal $C^{1,\alpha/2}$ -regularity for minimizers to the double obstacle problem with h^\pm as obstacles. The proof consists of several steps and an iteration argument. Here is a scheme of the steps:

Step 1) From Corollary 2.5 and Lemma 6.5 we already have some regularity, in particular $C^{1,\frac{\alpha}{2+\alpha}}$ -regularity for u and one-sided $C^{1,\frac{\alpha}{2+\alpha}}$ -estimates for h^\pm at points where $h^\pm = u$.

Step 2) Having $C^{1,\beta}$ -estimates for u and one-sided $C^{1,\beta}$ -estimates for h^\pm , we can apply Proposition 3.2 and Proposition 4.1, which will give

us a slight, let us say an ϵ , gain in the Hölder exponent for one-sided estimates in the x' -directions for h^\pm . That is, h^\pm satisfy one-sided $C^{1,\beta+\epsilon}$ -estimates in the x' -directions.

Step 3) Using that u solves the obstacle problem together with one-sided $C^{1,\beta+\epsilon}$ -estimates we can show that $u \in C^{1,\beta+\epsilon}$. In particular, we have gained an $\epsilon = \epsilon(\alpha, \beta)$ in regularity as compared to Corollary 2.5 and Lemma 6.5.

Step 4) We can iterate Step 2) and 3) to gain more regularity of u , but in order to fully utilize the ϵ gain in regularity we need to be able to control the $C^{1,\beta+\epsilon}$ -norm of u . As it turns out there is a constant ξ such that if $[a]_{C^\alpha} \leq \xi$ then we will get good control over the $C^{1,\beta+\epsilon}$ -norm of u . So we will rescale u and a by a factor τ , where τ is chosen such that $[a(\tau x)]_{C^\alpha} \leq \xi$. With this rescaling we can iterate Step 2) and 3) to gain another ϵ in the Hölder exponent *and also preserve the Hölder norm*.

Iterating Step 2) and 3) we will see that $u \in C^{1,\beta}$ for any $\beta < \alpha/2$, but *with a uniform bound on the $C^{1,\beta}$ -norm*. The Theorem follows.

In reality, the proof will be somewhat more involved as we have different regularity on different scales which will result in some technical issues.

With this strategy in mind let us turn to the proof of the Main Theorem.

5.2. Proof of the Main Theorem. Without loss of generality we may assume that $|\nabla u(0)| = |u(0)| = 0$ — ∇u exists by Corollary 2.5— and that $a(x) = e_n + b(x)$ with $|b(x)| \leq A|x|^\alpha$ in B_1 . If this is not true, we may subtract $u(0) + \nabla u(0) \cdot x$ from u and f , add $\nabla u(0)$ to a and rotate the coordinate system to obtain this situation.

We may also rescale $u_\tau(x) = \frac{u(\tau x)}{\tau}$ with $\tau = (\xi/A)^{1/\alpha}$ for a ξ depending only on α and n to be determined later. The scaled solution u_τ will then minimize the Dirichlet energy in B_{R_τ} with $R_\tau = 1/\tau$ and constraints h_τ^\pm solving

$$\pm |\nabla h_\tau^\pm - a_\tau|^2 = \pm 1,$$

where $a_\tau(x) = a(\tau x)$. We set $A_\tau := [a_\tau]_{C^\alpha(B_{1/\tau})} = \xi$.

From Corollary 2.5 we have $\text{osc}_{B_r} u_\tau \leq CA_\tau^{1/(2+\alpha)} r^{1+\alpha/(2+\alpha)}$ whenever $r \leq A_\tau^{1/2} \min(R_\tau^{(2+\alpha)/2}, R_\tau)$ and also —using Lemma 6.5— $\sup_{B_r} h_\tau^+ \leq CA_\tau^{1/(2+\alpha)} r^{1+\alpha/(2+\alpha)}$. That is enough to apply Lemma 6.6 and Proposition 4.1 with $\beta_0 = \alpha/(2+\alpha)$ and $r \leq \min(A_\tau^{1/2} R_\tau^{1/(1-\beta_0)}, R_\tau)$.

Let us assume, for now, that $r \leq R_\tau \leq A_\tau^{1/2} R_\tau^{1/(1-\beta_0)}$. Then we have that for every $x_n \in (0, A_\tau^{-\frac{1-\beta_0}{2}} r^{1-\beta_0})$ and some $p(x_n) = p$ in the super-differential of h^+ at $(0, x_n)$,

$$(79) \quad \sup_{x' \in B'_r} (h_\tau^+(x', x_n) - h_\tau^+(0, x_n) - p \cdot x') \leq C A_\tau^{\frac{1}{2} - \frac{\alpha\beta_0}{2(2+\alpha-2\beta_0)}} r^{1 + \frac{\alpha}{2+\alpha-2\beta_0}},$$

$$\text{if } r \leq \min\left(A_\tau^{\frac{6\beta_0+\alpha\beta_0-2-\alpha}{2(4\beta_0-\alpha)}}, A_\tau^{-\frac{\beta_0}{\alpha-2\beta_0}}, A_\tau^{\frac{\beta_0}{2}} R_\tau^{\frac{2+\alpha-2\beta_0}{2}}\right).$$

With

$$F(r) = C A_\tau^{\frac{1}{2} - \frac{\alpha\beta_0}{2(2+\alpha-2\beta_0)}} r^{1 + \frac{\alpha}{2+\alpha-2\beta_0}},$$

it follows from (79) that u satisfies the assumptions of Proposition 2.4, that is, there is a constant \tilde{C} such that

$$(80) \quad \text{osc}_{x \in B_r} u_\tau \leq \tilde{C} A_\tau^{\frac{1}{2} - \frac{\alpha\beta_0}{2(2+\alpha-2\beta_0)}} r^{1 + \frac{\alpha}{2+\alpha-2\beta_0}},$$

$$\text{if } r \leq \rho_\tau := \min\left(A_\tau^{1/2} R_\tau^{1/(1-\beta_0)}, A_\tau^{\frac{6\beta_0+\alpha\beta_0-2-\alpha}{2(4\beta_0-\alpha)}}, A_\tau^{-\frac{\beta_0}{\alpha-2\beta_0}}, A_\tau^{\frac{\beta_0}{2}} R_\tau^{\frac{2+\alpha-2\beta_0}{2}}\right).$$

The estimate (80) gives us better regularity for small r . We would want to iterate that statement. Let us rewrite the statement with $\beta_1 = \alpha/(2 + \alpha - 2\beta_0)$:

$$(81) \quad \text{osc}_{B_r} u_\tau \leq \left(\tilde{C} A_\tau^{\frac{(1-\beta_0)\beta_1}{2}}\right) A_\tau^{\frac{1-\beta_1}{2}} r^{1+\beta_1} \leq A_\tau^{\frac{1-\beta_1}{2}} r^{1+\beta_1};$$

here the last inequality is valid if A_τ is small enough, however the size of A_τ is independent of β_0 and β_1 (given that $1/2 \geq \beta_0, \beta_1 \geq \alpha/(2 + \alpha)$). As $A_\tau = \xi$ by our rescaling, let us choose ξ as the largest constant ≤ 1 such that the last inequality holds for all $1/2 \geq \beta_0, \beta_1 \geq \alpha/(2 + \alpha)$. Then ξ depends only on α and n .

We are going to apply Proposition 3.2 and then Proposition 4.1 again with $\beta = \beta_1$. It is natural to split the proof into the four cases

- (1) $R_\tau \geq 1$ (in which case $A \geq \xi$) and $R_\tau \leq A_\tau R_\tau^{\frac{1}{1-\beta_0}}$,
- (2) $R_\tau \leq 1$ (in which case $A \leq \xi$) and $R_\tau \geq A_\tau R_\tau^{\frac{1}{1-\beta_0}}$,
- (3) $R_\tau \geq 1$ (in which case $A \geq \xi$) and $R_\tau \geq A_\tau R_\tau^{\frac{1}{1-\beta_0}}$,
- (4) $R_\tau \leq 1$ (in which case $A \leq \xi$) and $R_\tau \leq A_\tau R_\tau^{\frac{1}{1-\beta_0}}$.

Here (1) and (2) relate to the first and second block of values of Q in Proposition 4.1, respectively. These cases are stable in the following sense: when β increases in Case (1) or (2), then $A_\tau R_\tau^{\frac{1}{1-\beta}}$ increases if $R_\tau \geq 1$ and decreases if $R_\tau \leq 1$. That means that we stay in the same Case (1) or (2), respectively for larger β . In Case (3) and (4) this is

no longer true, so we might have to use one of the blocks of Q for a finite number of iterations and then switch over to the other block.

We will finish the proof in Case 1 first.

Case (1) ($R_\tau \geq 1$ and $R_\tau \leq A_\tau^{1/2} R_\tau^{1/(1-\beta_0)}$): In this case we may iterate and get a $\beta_2 = \alpha/(2 + \alpha - 2\beta_1)$ such that (80) holds with β_2 replacing β_0 . We may iterate indefinitely to obtain an increasing sequence of β_j such that $\beta_j = \alpha/(2 + \alpha - 2\beta_{j-1})$, and (80) holds with β_j replacing β_0 for each $j \in \mathbb{N}$. It is easy to see that $\beta_j \rightarrow \alpha/2$ as $j \rightarrow \infty$. Using that $A_\tau = \xi$ is a constant depending only on α and n and that $R_\tau \geq 1$, it follows that

$$(82) \quad \text{osc}_{B_r} u_\tau \leq \tilde{C} A_\tau^{\frac{4-\alpha^2}{8}} r^{1+\frac{\alpha}{2}}$$

$$\text{for } r \leq \inf_j \left(\min \left(A_\tau^{\frac{6\beta_j + \alpha\beta_j - 2 - \alpha}{2(4\beta_j - \alpha)}}, A_\tau^{-\frac{\beta_j}{\alpha - 2\beta_j}}, A_\tau^{\frac{\beta_j}{2}} R_\tau^{\frac{2 + \alpha - 2\beta_j}{2}} \right) \right) \leq C_\alpha.$$

Rescaling back to u we obtain from (82) as well as the definition of τ that

$$(83) \quad \text{osc}_{B_r} u = \tau \text{osc}_{B_{r/\tau}} u_\tau \leq C_1 \tau^{-\alpha/2} r^{1+\alpha/2} \leq C_2 A_\tau^{1/2} r^{1+\alpha/2},$$

for $r \leq C_3 A_\tau^{-\frac{1}{\alpha}}$; here C_1, C_2 and C_3 are constants depending only on α and n .

When $r \geq C_3 A_\tau^{-\frac{1}{\alpha}}$ we obtain from $\beta_0 = \frac{\alpha}{2+\alpha}$ that

$$\text{osc}_{B_r} u \leq C_4 A_\tau^{\frac{1-\beta_0}{2}} r^{1+\beta_0} \leq C_5 A_\tau^{\frac{1}{2}} r^{1+\frac{\alpha}{2}},$$

where C_4 and C_5 are constants depending only on α and n . Combining the two estimates proves the Theorem in Case (1).

Case (2) ($R_\tau \leq 1$ and $R_\tau \geq A_\tau^{1/2} R_\tau^{1/(1-\beta_0)}$):

When $A_\tau^{1/2} R_\tau^{\frac{1}{1-\beta_0}} = \inf(A_\tau^{1/2} R_\tau^{\frac{1}{1-\beta_0}}, R_\tau)$, we have to use the second block of values for Q in Proposition 4.1. We deduce that

$$\sup_{B'_r(0, x_n)} |h^+(x) - h^+(0, x_n) - p \cdot x| \leq C A^{\frac{1}{2} - \frac{\alpha\beta_0}{2(2+\alpha-2\beta_0)}} r^{1 + \frac{\alpha}{2+\alpha-2\beta_0}}$$

$$\text{if } r \leq \min \left(A_\tau^{\frac{6\beta_0 + \alpha\beta_0 - 2 - \alpha}{2(4\beta_0 - \alpha)}}, A_\tau^{-\frac{\beta_0}{\alpha - 2\beta_0}}, A_\tau^{\frac{2+\alpha}{4}} R_\tau^{\frac{2+\alpha-2\beta_0}{2(1-\beta_0)}} \right).$$

Using Proposition 2.4 and iterating, we obtain that for β_j defined above,

$$(84) \quad \text{osc}_{x \in B_r} u_\tau \leq \tilde{C} A_\tau^{\frac{1}{2} - \frac{\alpha\beta_j}{2(2+\alpha-2\beta_j)}} r^{1 + \frac{\alpha}{2+\alpha-2\beta_j}}$$

$$\text{for } r \leq \min \left(A_\tau^{\frac{6\beta_j + \alpha\beta_j - 2 - \alpha}{2(4\beta_j - \alpha)}}, A_\tau^{-\frac{\beta_j}{\alpha - 2\beta_j}}, A_\tau^{\frac{2+\alpha}{4}} R_\tau^{\frac{2+\alpha-2\beta_j}{2(1-\beta_j)}} \right).$$

As in (81) we conclude that

$$(85) \quad \text{osc}_{x \in B_r} u_\tau \leq A_\tau^{\frac{1}{2} - \frac{\alpha\beta_j}{2(2+\alpha-2\beta_j)}} r^{1 + \frac{\alpha}{2+\alpha-2\beta_j}}$$

for $r \leq \min\left(A_\tau^{\frac{6\beta_j+\alpha\beta_j-2-\alpha}{2(4\beta_j-\alpha)}}, A_\tau^{-\frac{\beta_j}{\alpha-2\beta_j}}, A_\tau^{\frac{2+\alpha}{4}} R_\tau^{\frac{2+\alpha-2\beta_j}{2(1-\beta_j)}}\right)$.

Sending j to infinity and using that $R_\tau < 1$ and that $\frac{2+\alpha-2\beta_j}{2(1-\beta_j)}$ is decreasing in β_j we obtain that

$$\text{osc}_{x' \in B_r} u_\tau \leq C_6 A_\tau^{\frac{1}{2} - \frac{\alpha^2}{8}} r^{1 + \frac{\alpha}{2}}$$

for $r \leq A_\tau^{\frac{2+\alpha}{4}} R_\tau^{\frac{2}{2-\alpha}}$, where C_6 is a constant depending only on α and n .

Using Proposition 2.4 as well as the second case in the second block in Proposition 4.1 in iteration, we also see that

$$\text{osc}_{x' \in B'_r} u_\tau \leq C_7 R_\tau^{-1} r^2$$

for $r \geq A_\tau^{\frac{2+\alpha}{4}} R_\tau^{\frac{2}{2-\alpha}}$, where C_7 is a constant depending only on α and n .

Scaling back and using that A_τ is a constant depending only on α and n we obtain that

$$\text{osc} u \leq \begin{cases} C_8 A_\tau^{\frac{1}{2}} r^{1 + \frac{\alpha}{2}} & \text{if } r \leq C_{10} A_\tau^{\frac{2}{2-\alpha}}, \\ C_9 r^2 & \text{if } r \geq C_\alpha A_\tau^{\frac{2}{2-\alpha}}, \end{cases}$$

where C_8, C_9 and C_{10} are constants depending only on α and n . This concludes the proof in Case (2).

Case (3) ($R_\tau \geq 1$ and $R_\tau \geq A_\tau^{1/2} R_\tau^{1/(1-\beta_0)}$): We start as in Case (2) and deduce that for all j such that

$$(86) \quad R_\tau \geq A_\tau^{\frac{1}{2}} R_\tau^{\frac{1}{1-\beta_j-1}},$$

we have

$$\sup_{B'_r(0, x_n)} |h^+(x) - h^+(0, x_n) - p \cdot x| \leq C A_\tau^{\frac{1}{2} - \frac{\alpha\beta_j}{2(2+\alpha-2\beta_j)}} r^{1 + \frac{\alpha}{2+\alpha-2\beta_j}}$$

for $r \leq \min\left(A_\tau^{\frac{6\beta_j+\alpha\beta_j-2-\alpha}{2(4\beta_j-\alpha)}}, A_\tau^{-\frac{\beta_j}{\alpha-2\beta_j}}, A_\tau^{\frac{2+\alpha}{4}} R_\tau^{\frac{2+\alpha-2\beta_j}{2(1-\beta_j)}}\right)$.

If (86) holds for all j then we are done as in Case (2). If (86) is not true then there is a largest j_0 such that the inequality holds for

all $j \leq j_0$. It follows that $R_\tau \geq 1$ and $R_\tau \leq A_\tau^{1/2} R_\tau^{\frac{1}{1-\beta_{j_0}}}$. These are the assumptions in the iteration in Case (1), so we may proceed as in Case (1) and obtain the statement of the Theorem.

Case (4) ($R_\tau \leq 1$ and $R_\tau \leq A_\tau^{1/2} R_\tau^{1/(1-\beta_0)}$): We start as in Case (1) and deduce that as long as $R_\tau \leq A_\tau^{1/2} R_\tau^{1/(1-\beta_{j-1})}$,

$$(87) \quad \text{osc}_{x \in B_r} u_\tau \leq A_\tau^{\frac{1}{2} - \frac{\alpha\beta_j}{2(2+\alpha-2\beta_j)}} r^{1 + \frac{\alpha}{2+\alpha-2\beta_j}},$$

for $r \leq \min\left(A_\tau^{1/2} R_\tau^{1/(1-\beta_j)}, A_\tau^{\frac{6\beta_j + \alpha\beta_j - 2 - \alpha}{2(4\beta_j - \alpha)}}, A_\tau^{-\frac{\beta_j}{\alpha-2\beta_j}}, A_\tau^{\frac{\beta_j}{2}} R_\tau^{\frac{2+\alpha-2\beta_j}{2}}\right)$.

If $R_\tau \leq A_\tau^{1/2} R_\tau^{1/(1-\beta_{j-1})}$ for all j then we are done. If there is a j_0 such that $R_\tau \geq A_\tau^{1/2} R_\tau^{1/(1-\beta_{j_0})}$ then we are in the situation of Case (2), so we may proceed as in Case (2) and obtain the statement of the Theorem.

This finishes the proof. \square

6. APPENDIX: REMARKS ON VISCOSITY SOLUTIONS FOR HAMILTON-JACOBI EQUATIONS.

In this appendix we will remind ourselves of some basic properties of viscosity solutions for Hamilton-Jacobi equations. Most of the results in the appendix can be found in [?]. However the exposition in [?] is quite sketchy at points and some of the results that we need are not explicitly proved. We will therefore, for the readers convenience, supply some details, without claiming any originality. For the original and classical papers on the theory of viscosity solutions we direct the reader to [?], [?] and [?]. First we will state the definition of viscosity solutions for first order Hamilton-Jacobi equations.

Definition 6.1. We say that $u \in C^0(\Omega)$ is a viscosity solution to

$$H(x, u, \nabla u) = f(x) \text{ in } \Omega,$$

if for every $x^0 \in \Omega$ and $\phi \in C^1(B_r(x^0))$ such that $u(x^0) = \phi(x^0)$ the following holds:

- (1) if $u(x^0) = \phi(x^0)$ and $u(x) - \phi(x)$ has a local maximum at x^0 , then $H(x^0, \phi(x^0), \nabla \phi(x^0)) \leq f(x^0)$,
- (2) if $u(x^0) = \phi(x^0)$ and $u(x) - \phi(x)$ has a local minimum at x^0 , then $H(x^0, \phi(x^0), \nabla \phi(x^0)) \geq f(x^0)$.

This definition turns out to be the right one for first order Hamilton-Jacobi equations in the sense that it provides strong existence and uniqueness results.

Viscosity solutions can be obtained by the vanishing viscosity method. That method also implies one-sided estimates for the second derivatives of solutions to smooth convex first order Hamilton-Jacobi equations which we will prove next.

Lemma 6.2. *Let $u \in C^0(B_R \times (-R, 0))$ be a viscosity solution to*

$$\frac{\partial u}{\partial t} + H(x, u, \nabla u) = m(x)$$

such that $H \in C^2$ and

$$p \cdot D_p^2 H(x, a, p) \cdot p \geq c|p|^2 \text{ for every } (x, a, p).$$

Moreover we assume that $m \in C^{1,1}$. Then the distributional second derivatives satisfy

$$\frac{\partial^2 u}{\partial x_i^2} \leq \frac{C_0}{R} \text{ in } B_{R/2} \times (-R/2, 0)$$

for all spatial directions x_i . The constant C_0 depends only on $\|D^2 m\|_{L^\infty(B_R)}$, $\sup_{B_R} |\nabla u|$ and H .

A similar statement holds for the time independent case, that is if $u \in C^0(B_R)$ is a viscosity solution to

$$H(x, u, \nabla u) = m(x)$$

such that $H \in C^2$ and

$$p \cdot D_p^2 H(x, a, p) \cdot p \geq c|p|^2 \text{ for every } (x, a, p)$$

and $m \in C^{1,1}$, then

$$\frac{\partial^2 u}{\partial x_i^2} \leq \frac{C_0}{R} \text{ in } B_{R/2}$$

for all spatial directions x_i . The constant C_0 depends only on $\|D^2 m\|_{L^\infty(B_R)}$, $\sup_{B_R} |\nabla u|$ and H .

Proof: We will only prove the first statement for parabolic Hamilton-Jacobi equations. The proof for elliptic Hamilton-Jacobi equations is similar. Alternatively the elliptic proof may be derived by considering t to be a dummy variable.

Moreover, as the proof is well known and included only for the sake of completeness, we will consider the slightly simpler case $H(x, a, p) = H(p)$. The general case can be handled similarly.

Rescaling $u(Rx, Rt)/R$ if necessary we may assume that $R = 1$. Let $\chi \in C^\infty(B_1 \times (-1, 0))$ be a non negative function such that $\chi = 1$ in $B_{1/2} \times (-1/2, 0)$ and $\chi = 0$ close to $\{t = -1\}$ and $\partial B_1 \times (-1, 1)$. Furthermore we may assume that $|\nabla \chi|, |\Delta \chi|, |\nabla \chi|^2/|\chi|, |\partial \chi / \partial t| \leq C$ in the support of χ .

We are going to argue by the method of vanishing viscosity. That is, we will approximate $\frac{\partial u}{\partial t} + H(\nabla u) = m(x)$ by the equation

$$(88) \quad -\epsilon \Delta u + \frac{\partial u}{\partial t} + H(\nabla u) = m(x)$$

in order to deduce the desired estimate independent of $\epsilon > 0$. The Lemma follows from uniform convergence by letting $\epsilon \rightarrow 0$.

Differentiating equation (88) twice in the direction x_i we get

$$-\epsilon \Delta u_{ii} + \frac{\partial u_{ii}}{\partial t} + H'_p(\nabla u) \cdot \nabla u_{ii} + (\nabla u_i \cdot H''_{pp}) \cdot \nabla u_i = m_{ii}(x).$$

Writing $w = \chi u_{ii}$ we see that

$$\begin{aligned} -\epsilon \Delta w &= -\chi \frac{\partial u_{ii}}{\partial t} - \chi (\nabla u_i \cdot H''_{pp}) \cdot \nabla u_i \\ &\quad - \chi H'_p(\nabla u) \cdot \nabla u_{ii} + \chi m_{ii} - 2\epsilon \nabla \chi \cdot \nabla u_{ii} - \epsilon u_{ii} \Delta \chi. \end{aligned}$$

At a point (x^0, t_0) where w attains its positive supremum we have $-\Delta w \geq 0$, $\nabla w = 0$ and $\partial w / \partial t \geq 0$. The last two conditions are equivalent to

$$\begin{aligned} \nabla u_{ii}(x^0, t_0) &= -\frac{u_{ii}(x^0, t_0) \nabla \chi(x^0, t_0)}{\chi(x^0, t_0)}, \\ \chi(x^0, t_0) \frac{\partial u_{ii}(x^0, t_0)}{\partial t} &\geq -u(x^0, t_0) \frac{\partial \chi(x^0, t_0)}{\partial t}. \end{aligned}$$

Using this together with the convexity assumption on H , we end up with

$$0 \leq u_{ii} \frac{\partial \chi}{\partial t} - c\chi |\nabla u_i|^2 + u_{ii} H'_p(\nabla u) \cdot \nabla \chi + \chi m_{ii} + 2\epsilon \frac{|\nabla \chi|^2}{\chi} u_{ii} - \epsilon u_{ii} \Delta \chi$$

at (x^0, t_0) . Rearranging terms we get

$$c\chi |\nabla u_i|^2 \leq \left(\frac{\partial \chi}{\partial t} + H'_p(\nabla u) \cdot \nabla \chi + 2\epsilon \frac{|\nabla \chi|^2}{\chi} - \epsilon \Delta \chi \right) u_{ii} + m_{ii} \chi.$$

Using that —by our choice of χ and assumptions on H — the terms in the parenthesis may be estimated by a constant C depending only on $\sup_{B_R} |\nabla u|$ and H , and observing that the final term $m_{ii} \chi$ is also bounded by the assumption $m \in C^{1,1}$, we obtain that at the point where w attains its supremum,

$$c\chi |\nabla u_i|^2 \leq C|u_{ii}| + C,$$

where C depends on $\sup_{B_1 \times (-1,0)} m_{ii}$, $\sup_{B_1 \times (-1,0)} |\nabla u|$ and $\sup |H'_p(\nabla u)|$. Multiplying both sides by χ implies that $|w| \leq C$. Which in terms of u_{ii} becomes

$$\sup_{B_{1/2} \times (-1/2,0)} u_{ii} \leq \sup_{B_1 \times (-1,0)} \chi u_{ii} \leq C.$$

□

Later on we will also need good stability estimates for solutions, proved in the next two Lemmas.

Lemma 6.3. *Let $H \in C^2$ and let $u \in C^{0,1}(\overline{B_R})$ and $v \in C^{0,1}(\overline{B_R})$ be viscosity solutions in B_R to*

$$H(x, \nabla u) = n(x)$$

and

$$H(x, \nabla v) = m(x)$$

where

$$p \cdot D_p^2 H(x, p) \cdot p \geq c|p|^2 \text{ for every } (x, p),$$

$m \geq n \geq \lambda > 0$, $u \leq v$ on ∂B_R and $\|u\|_{L^\infty(B_R)}, \|v\|_{L^\infty(B_R)} \leq K$. Then

$$u \leq v \leq u + C(\lambda, K)R \max \left(\sup_{B_R} (m - n), \sup_{\partial B_R} (v - u) \right).$$

Proof: It is sufficient to prove the Lemma when $\sup_{\partial B_R} (m - n)$ and $\sup_{B_R} (v - u)$ are small. By a rescaling we may assume that $R = 1$.

Claim 1: *Let*

$$w_1 = e^{K+1} - e^{-u}.$$

Then there exists a function $F(x, a, p)$, convex in p , such that

$$(89) \quad \frac{\partial F(x, a, p)}{\partial a} \geq \lambda,$$

and in the viscosity sense

$$F(x, w_1, \nabla w_1) = 0.$$

Moreover $w_2 = e^{K+1} - e^{-v}$ is a viscosity solution to

$$F(x, w_2, \nabla w_2) = (e^{K+1} - w_2)(m - n).$$

Proof of Claim 1: A simple calculation shows that

$$(e^{K+1} - w_1)H(x, \nabla w_1/(e^{K+1} - w_1)) = (e^{K+1} - w_1)m,$$

so if we denote

$$F(x, a, p) = (e^{K+1} - a)H\left(x, \frac{p}{e^{K+1} - a}\right) - (e^{K+1} - a)m,$$

the first statement in the claim follows. That F is convex in p follows from the convexity of H . A direct calculation yields

$$\frac{\partial F(x, a, p)}{\partial a} = -H\left(x, \frac{p}{e^{K+1} - a}\right) + H_p\left(x, \frac{p}{e^{K+1} - a}\right) \cdot \frac{p}{e^{K+1} - a} + m \geq m \geq \lambda$$

where we have used convexity of H . That

$$F(x, w_2, \nabla w_2) = (e^{K+1} - w_2)(m - n).$$

follows from a simple calculation. The claim follows.

Next we observe that we may assume that $\Delta w_1 \leq C_\Delta$ for some constant C_Δ : we may regularize F so it becomes C^2 . Denote the regularized version of F by F_τ and let w_1^τ solve

$$F_\tau(x, w_1^\tau, \nabla w_1^\tau) = 0,$$

then by Lemma 6.2, $\Delta w_1^\tau \leq C_\tau$. We also know that $w_1^\tau \in C^\alpha$ uniformly for all $\tau > 0$, thus we find a subsequence $\tau_j \rightarrow 0$ such that $w_1^{\tau_j} \rightarrow w_1^0$ uniformly. Using uniqueness results for viscosity solutions it is easy to see that $w_1^0 = w_1$. So if we can show the Lemma for each $w_1^{\tau_j}$ the result follows for w_1 . Thus we may assume that $\Delta w_1 \leq C_\Delta$ as long as our final estimate does not depend on C_Δ . We will make several more regularizations in what follows. In order to simplify notation, we will by the just explained argument assume that $\Delta w_1 \leq C_\Delta$, and continue to work with w_1 and not with $w_1^{\tau_j}$.

Next, we apply a standard convolution type regularization to w_1 and w_2 , that is we denote

$$w_i^\delta = \int w_i(y) \phi_\delta(x - y) dy,$$

where ϕ_δ is a standard mollifier. Then

$$F(x, w_1^\delta, \nabla w_1^\delta) = h_1^\delta$$

$$\text{and } F(x, w_2^\delta, \nabla w_2^\delta) = (e^{K+1} - w_2)(m - n) + h_2^\delta,$$

where $h_i^\delta \rightarrow 0$ uniformly in L^p for all $p \in [1, +\infty)$ as $\delta \rightarrow 0$. To do this regularization we need to assume that u and v are solutions in $B_{1+\delta}$ or prove the result in $B_{1-\delta}$. However as $\delta \rightarrow 0$ this will not make any difference so we may as well ignore this slight complication.

Finally we will denote by $w_2^{\delta, \epsilon}$ the solution to

$$-\epsilon \Delta w_2^{\delta, \epsilon} + F(x, w_2^{\delta, \epsilon}, \nabla w_2^{\delta, \epsilon}) = h_2^\delta, w_2^{\delta, \epsilon} = w_2^\delta \text{ on } \partial B_1.$$

With these regularizations we end up with

$$\begin{aligned} & -\epsilon \Delta (w_2^{\delta, \epsilon} - w_1^\delta) + F(x, w_2^{\delta, \epsilon}, \nabla w_2^{\delta, \epsilon}) - F(x, w_1^\delta, \nabla w_1^\delta) \\ & \leq (e^{K+1} - w_2)(m - n) + \epsilon C_\Delta + h_2^\delta - h_1^\delta. \end{aligned}$$

Next, in order to linearize we define

$$B(x) = \int_0^1 F_{,p}(x, w_1^\delta, t \nabla w_2^{\delta, \epsilon} - (1 - t) \nabla w_1^\delta) dt.$$

Then, taking equation (89) into consideration,

$$-\epsilon \Delta (w_2^{\delta, \epsilon} - w_1^\delta) + B(x) \cdot (\nabla w_2^{\delta, \epsilon} - \nabla w_1^\delta) + \lambda (w_2^{\delta, \epsilon} - w_1^\delta)$$

$$\leq (e^{K+1} - w_2)(m - n) + \epsilon C_\Delta + h_2^\delta - h_1^\delta.$$

Now let $f^{\delta,\epsilon}$ be the solution to

$$-\epsilon \Delta f^{\delta,\epsilon} + B(x) \cdot \nabla f^{\delta,\epsilon} + \lambda f^{\delta,\epsilon} = (e^{K+1} - w_2)(m - n) + C_\Delta \epsilon$$

with boundary values $f^{\delta,\epsilon} = \sup_{\partial B_1} (w_2^{\delta,\epsilon} - w_1^{\delta,\epsilon})$ and $g^{\delta,\epsilon}$ be the solution to

$$-\epsilon \Delta g^{\delta,\epsilon} + B(x) \cdot \nabla g^{\delta,\epsilon} + \lambda g^{\delta,\epsilon} = h_2^\delta - h_1^\delta,$$

with boundary values $g^{\delta,\epsilon} = 0$ on ∂B_1 . By the comparison principle

$$(90) \quad w_2^{\delta,\epsilon} - w_1^{\delta,\epsilon} \leq f^{\delta,\epsilon} + g^{\delta,\epsilon}.$$

Since $h_2^\delta - h_1^\delta \rightarrow 0$ in L^p for any $p < +\infty$ as $\delta \rightarrow 0$ we see that $\sup_{B_1} |g^{\delta,\epsilon}| \rightarrow 0$ as $\delta \rightarrow 0$. Next we notice that where $f^{\delta,\epsilon}$ attains its supremum we have $|\nabla f^{\delta,\epsilon}| = 0$ and $\Delta f^{\delta,\epsilon} \leq 0$. It follows that

$$\sup_{B_1} f^{\delta,\epsilon} \leq \max \left(\sup_{\partial B_1} f^{\delta,\epsilon}, \sup_{B_1} \frac{1}{\lambda} (e^{K+1} - w_2)(m - n) + \frac{C\epsilon}{\lambda} \right).$$

Letting first $\delta \rightarrow 0$ and then $\epsilon \rightarrow 0$ we see that equation (90) implies that

$$w_2 - w_1 \leq \max \left(\sup_{\partial B_1} (w_2 - w_1), \sup_{B_1} \frac{1}{\lambda} (e^{K+1} - w_2)(m - n) \right).$$

Writing this last inequality in terms of u and v we get

$$e^{-u} - e^{-v} \leq \max \left(\sup_{\partial B_1} (e^{-u} - e^{-v}), \sup_{B_1} \frac{e^{-v}}{\lambda} (m - n) \right).$$

Thus

$$1 - e^{u-v} \leq e^{2K} \max \left(\sup_{\partial B_1} (1 - e^{u-v}), \sup_{B_1} \frac{1}{\lambda} (m - n) \right).$$

As pointed out in the beginning of the proof it is enough to show the Lemma when $\sup_{B_1} (m - n)$ and $\sup_{\partial B_1} (v - u)$ are small. Using $\xi/2 \leq 1 - e^{-\xi} \leq 2\xi$ for small ξ , we end up with

$$\sup_{B_1} (v - u) \leq 2e^{2K} \max \left(\frac{1}{\lambda} \sup_{B_1} (m - n), 2 \sup_{\partial B_1} (v - u) \right).$$

□

Next we need a comparison estimate for parabolic Hamilton-Jacobi equations. The proof is very similar to the proof of Lemma 6.3 so we will omit it here with a reference to [?].

Lemma 6.4. *Let $H \in C^2$ and let $u \in C^{0,1}(\overline{B_R} \times (-R, 0))$ and $v \in C^{0,1}(\overline{B_R} \times (-R, 0))$ be viscosity solutions in $B_R \times (-R, 0)$ to*

$$\frac{\partial u}{\partial t} + H(\nabla u) = m(x)$$

and

$$\frac{\partial v}{\partial t} + H(\nabla v) = n(x),$$

where

$$p \cdot D^2 H(p) \cdot p \geq c|p|^2 \text{ for every } p,$$

$n > m$, $u = v$ on ∂B_R for $t \in (-R, 0)$ and $u(x, -R) = v(x, -R)$ for $x \in B_R$.

Then

$$u \leq v \leq u + CR \sup_{B_R \times (-R^2, 0)} (n - m).$$

Proof: For a discussion of a proof see [?]. \square

From the regularity results in Lemma 6.2 and 6.3 we can easily deduce some elementary, non-optimal, one-sided estimates for solutions to Hamilton-Jacobi equations, even when the data is not C^2 . This incidentally provides our starting regularity for a bootstrap argument which in turn yields optimal regularity.

Lemma 6.5. *Let h be a viscosity solution to $|\nabla h - a|^2 = 1$ in B_R , $a \in C^\alpha(\overline{B_R}; \mathbb{R}^n)$ and $[a(x)]_{C^\alpha(\overline{B_R})} \leq A$. Then for any $x^0 \in B_{R/2}$ the super-differential of h at x^0 is not empty, and for any p in the super-differential of h at x^0 we have*

(1)

$$\sup_{B_r(x^0)} (h(x) - p \cdot (x - x^0) - h(x^0)) \leq CA^{1/(2+\alpha)} r^{1+\alpha/(2+\alpha)}$$

for $r \leq \min(A^{1/2}R^{(2+\alpha)/2}, R)$,

(2) and

$$\sup_{B_r(x^0)} (h(x) - p \cdot (x - x^0) - h(x^0)) \leq C \frac{r^2}{R}$$

for $A^{-1/\alpha} \geq r \geq A^{1/2}R^{(2+\alpha)/2}$.

The constant C depends only on n .

Proof: We may assume that $h(0) = 0$.

First we notice that —as $r \leq R$ — if $r \geq A^{-1/\alpha}$ then $r \leq A^{1/2}R^{(2+\alpha)/2}$. Also, if $r \geq A^{-1/\alpha}$ then

$$1 = \sup_{B_r} |\nabla h - a| \geq \sup_{B_r} |\nabla h| - \sup_{B_r} |a| = \sup_{B_r} |\nabla h| - Ar^\alpha.$$

Using that $Ar^\alpha \geq 1$, it follows that

$$\sup_{B_r} |h| \leq r \sup_{B_r} |\nabla h| \leq r(1 + Ar^\alpha) \leq CA^{1/(2+\alpha)} r^{1+\alpha/(2+\alpha)}.$$

This shows that if $r \geq A^{-1/\alpha}$ then both the assumption and the conclusion in (1) hold.

We therefore only have to show the Lemma when $r \leq A^{-1/\alpha}$. We do this by a barrier type argument. We may assume that $a(0) = 0$. We also define the barrier w as the solution to

$$\begin{aligned} |\nabla w|^2 &= 1 + (2A\delta^\alpha + A^2\delta^{2\alpha}) && \text{in } B_\delta(x^0), \\ w &= h && \text{on } \partial B_\delta(x^0), \end{aligned}$$

where δ is to be determined later. Then $|\nabla w| \leq \sqrt{1 + (2A\delta^\alpha + A^2\delta^{2\alpha})} \leq 1 + (2A\delta^\alpha + A^2\delta^{2\alpha})$ and thus

$$\begin{aligned} |\nabla w - a| &\geq |\nabla w| - |a| \geq \sqrt{1 + (2A\delta^\alpha + A^2\delta^{2\alpha})} - A\delta^\alpha \\ &= 1 + \left(\sqrt{1 + (2A\delta^\alpha + A^2\delta^{2\alpha})} - (1 + A\delta^\alpha) \right) = 1, \end{aligned}$$

so w is a super-solution to $|\nabla h - a|^2 = 1$. Similarly,

$$|\nabla w - a|^2 \leq 1 + CA\delta^\alpha(1 + A\delta^\alpha).$$

We may thus use the comparison estimate for Hamilton-Jacobi equations (Lemma 6.3) and derive

$$\begin{aligned} h &\leq w \leq w(x^0) + p \cdot (x - x^0) + \frac{C|x - x^0|^2}{\delta} \\ &\leq h(x^0) + p \cdot (x - x^0) + \frac{C|x - x^0|^2}{\delta} + CA\delta^{1+\alpha}(1 + A\delta^\alpha) \end{aligned}$$

for p in the super-differential of w . Rearranging the terms and taking the supremum in $B_r(x^0)$ for some $r \leq \delta$ yields

$$(91) \quad \sup_{B_r(x^0)} \left(h(x) - p \cdot (x - x^0) - h(x^0) \right) \leq C \left(\frac{r^2}{\delta} + A\delta^{1+\alpha}(1 + A\delta^\alpha) \right).$$

We want to find the right balance between r and δ that optimises (91). It is convenient to divide the last part of the proof into two cases:

Case 1: When $r \leq \min(A^{1/2}R^{\frac{2+\alpha}{2}}, A^{-\frac{1}{\alpha}})$, we use $\delta = A^{-\frac{1}{2+\alpha}}r^{\frac{2}{2+\alpha}}$ in (91) and deduce that —using $r \leq A^{-1/\alpha}$ which implies that $A\delta^\alpha = A^{\frac{2}{2+\alpha}}r^{\frac{2\alpha}{2+\alpha}} \leq 1$ —

$$\begin{aligned} &\sup_{B_r(x^0)} \left(h(x) - p \cdot (x - x^0) - h(x^0) \right) \\ &\leq C \left(A^{\frac{1}{2+\alpha}} r^{1+\frac{\alpha}{2+\alpha}} + A\delta^{1+\alpha}(1 + A\delta^\alpha) \right) \leq CA^{\frac{1}{2+\alpha}} r^{1+\frac{\alpha}{2+\alpha}}. \end{aligned}$$

From the definition of r and δ it is easy to check that $r \leq \delta \leq R$.

Case 2: Next, if $A^{1/2}R^{\frac{2+\alpha}{2}} \leq r \leq A^{-1/\alpha}$ then we use $\delta = R$ in (91) and deduce that —using $A^{1/2}R^{\frac{2+\alpha}{2}} \leq r$, $A^{1/2}R^{\frac{2+\alpha}{2}} \leq r \leq A^{-1/\alpha}$ and $AR^\alpha \leq 1$ —

$$\sup_{B_r(x^0)} (h(x) - p \cdot (x - x^0) - h(x^0)) \leq C \left(\frac{r^2}{R} + AR^{1+\alpha} (1 + AR^\alpha) \right) \leq C \frac{r^2}{R}.$$

Observe that this interval is empty unless $R \leq A^{-1/\alpha}$. \square

We end this appendix with a lemma reminiscent of a Whitney extension theorem, which will be useful in the main text.

Lemma 6.6. *Let $h \in C^{0,1}(\overline{B_1})$ and assume that h satisfies the one-sided $C^{1,\beta}$ -estimate*

$$(92) \quad h(x^0 + x) \leq h(x^0) + p_{x^0} \cdot (x - x^0) + C_0|x|^{1+\beta}$$

for every p_{x^0} in the super-differential of h at x^0 and every $x^0 \in B_1$. Moreover assume that $h(x) \geq -C_1|x|^{1+\beta}$ and that $h(0) = 0$. Then

$$(93) \quad |p_x| \leq E(\beta)(C_0 + C_1)|x|^\beta \text{ for } x \in B_{1/2};$$

here $E(\beta)$ depends continuously on β for $\beta \in (0, 1)$.

Proof: Take $y \in B_r$ and let $p_y = p$ be in the super-differential of h at y . As h is semi-concave by (92), we know that the super-differential of h is non-empty for every x^0 . Notice that, due to (93), the super-differential of h at the origin contains only the zero vector.

Therefore $h(y) \leq C_0r^{1+\beta}$. Now consider $z = y - \epsilon p$ with

$$\epsilon = (\delta|p|^{1-\beta})^{\frac{1}{\beta}}$$

for some small constant δ . By one-sided estimates from above and below we have

$$\begin{aligned} & -C_\beta C_1 (r^{1+\beta} + \epsilon^{1+\beta} |p|^{1+\beta}) \\ & \leq -C_1 |y - \epsilon p|^{1+\beta} \leq h(z) \leq h(y) - \epsilon |p|^2 + C_0 \epsilon^{1+\beta} |p|^{1+\beta}, \end{aligned}$$

where C_β is a constant depending only on β and n . Reordering terms and using that $h(y) \leq C_0r^{1+\beta}$ yields

$$(\delta^{1/\beta} - (C_\beta C_1 + C_0)\delta^{(1+\beta)/\beta})|p|^{\frac{1+\beta}{\beta}} \leq (C_0 + C_\beta C_1)r^{1+\beta}.$$

Choosing $\delta = (2C_0 + 2C_\beta C_1)^{-1}$, the previous inequality becomes, for some C depending only on β ,

$$|p|^{(1+\beta)/\beta} \leq C(C_1 + C_0)^{(1+\beta)/\beta} r^{1+\beta}.$$

Taking both sides to the power of $\beta/(1 + \beta)$, the lemma follows. \square

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